A nonlinear nonautonomous delay differential inequality for dissipativity of Lotka-Volterra functional differential equations

Liguang Xu[a,*], Danhua He[b]

[a] Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou, 310023, PR China
[b] Department of Mathematics, Zhejiang International Studies University, Hangzhou, 310012, PR China
*Corresponding Author
E-mail: xlg132@126.com

Abstract

In this paper, a Lotka-Volterra functional differential equation is considered. By establishing a nonlinear nonautonomous delay differential inequality and using a generalized Barbălat's lemma, we obtain some new sufficient conditions ensuring the dissipativity of the Lotka-Volterra functional differential equation.

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1 Introduction

Let H be a real or complex Hilbert space with the inner product $< \cdot, \cdot >$ and the corresponding norm $\|\cdot\|$, X be a dense continuously imbedded subspace of H. For any given closed interval $I \subset R$, let the symbol $C_X(I)$ denote a Banach space consisting of all continuous mappings $x : I \to X$, on which the norm is defined by $\|x\|_\infty = \max_{t \in I} \|x(t)\|$. Consider the initial value problem in Lotka-Volterra functional differential equations

$$\begin{cases} y'(t) = g(t, y(t), y(\cdot)) = \eta(t, y(t)) f(t, y(t), y(\cdot)), \ t \geq 0, \\ y(t) = \varphi(t), \ -\tau \leq t \leq 0, \end{cases} \tag{1}$$

where τ is a positive constant, $\varphi \in C_X[-\tau, 0]$ is a given intial function, $\eta : [0, +\infty) \times X \to H$ is a nonnegative continuous function, $f : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to H$, and $g : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to H$ is a given locally Lipschitz continuous mapping satisfying

$$\begin{aligned} &2\Re < u, g(t, u, \psi(\cdot)) > \leq \eta(t, u)[\gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{t-\mu_2(t) \leq \theta \leq t-\mu_1(t)} \|\psi(\theta)\|^2], \\ &u \in X, \psi \in C_X[-\tau, +\infty), t \in [0, +\infty), \end{aligned} \tag{2}$$

where the functions $\mu_1(t)$ and $\mu_2(t)$ are assumed to satisfy

$$0 \leq \mu_1(t) \leq \mu_2(t) \leq t + \tau, \forall t \in [0, +\infty), \tag{3}$$

$\alpha(t)$ and $\beta(t)$ are continuous functions and $\gamma(t)$ is a bounded continuous functions on the interval $[0, +\infty)$.

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Recently, Wen, Yu and Wang [1] discussed the dissipativity of (1) with $\eta(t, y(t)) \equiv 1$. They established the generalized Halanay inequality and obtained the dissipativity results of (1) with $\eta(t, y(t)) \equiv 1$. In this paper, we will improve the inequality in [1] such that it is effective for (1). By establishing a nonlinear nonautonomous delay differential inequality and using a generalized Barbălat's lemma, we obtain some new sufficient conditions ensuring the dissipativity of (1).

2 Nonlinear delay differential inequality

Theorem 2.1. *If $y(t) \geq 0$ is a differentiable function defined on $(-\infty, +\infty)$, and*

$$\begin{cases} u'(t) \leq \eta(t, y(t))[\gamma(t) + \alpha(t)u(t) + \beta(t)\sup_{t-\tau(t)\leq\theta\leq t} u(\theta)], t \geq t_0, \\ y(t) = \psi(t), t \leq t_0, \end{cases} \tag{4}$$

where $u(t) = \|y(t)\|^2$, and $\psi(t)$ is bounded and continuous for $t \leq t_0$, continuous functions $\gamma(t) \geq 0$, $\beta(t) \geq 0$ and $\alpha(t) \leq 0$ for $t \in [t_0, +\infty)$, $\tau(t) \geq 0$, $\eta : [0, +\infty) \times X \to H$ is a nonnegative continuous function, and if there exists $\sigma > 0$ such that

$$\alpha(t) + \beta(t) \leq -\sigma < 0 \text{ for } t \geq t_0, \tag{5}$$

then we have
(i)

$$u(t) \leq \frac{\gamma^*}{\sigma} + G, t \geq t_0. \tag{6}$$

(ii)

$$u(t) \leq \frac{\gamma^*}{\sigma} + Ge^{-\mu^* \int_{t_0}^{t} \eta(s,y(s))ds}, t \geq t_0, \tag{7}$$

where $G = \sup_{-\infty<\theta\leq t_0} \|\psi(\theta)\|^2$, $\gamma^* = \sup_{t_0\leq t<+\infty} \gamma(t)$, and $\mu^* \geq 0$ is defined as

$$\mu^* = \inf_{t\geq t_0} \{\mu(t) : \mu(t) + \alpha(t) + \beta(t)e^{h\mu(t)\tau(t)} = 0\}, \tag{8}$$

where

$$h = \sup_{t\geq t_0} \max_{(s,u)\in[t-\tau,t]\times[0,\frac{\gamma^*}{\sigma}+G]} \eta(t, y) < \infty. \tag{9}$$

Proof. (i): We at first shall prove that for any positive constant ε,

$$u(t) \leq \frac{\gamma^* + \varepsilon}{\sigma} + G, t \geq t_0. \tag{10}$$

If (10) does not hold, then there exists $t_1 > t_0$ such that

$$u(t_1) = \frac{\gamma^* + \varepsilon}{\sigma} + G, u'(t_1) > 0, u(t) \leq \frac{\gamma^* + \varepsilon}{\sigma} + G, t \in (-\infty, t_1]. \tag{11}$$

Using (4), (5) and (11), we obtain that

$$
\begin{aligned}
u'(t_1) &\le \eta(t_1, y(t_1))[\gamma(t_1) + \alpha(t_1)u(t_1) + \beta(t_1) \sup_{t_1-\tau(t_1)\le\theta\le t_1} u(\theta)] \\
&\le \eta(t_1, y(t_1))[\gamma^* + \alpha(t_1)(\frac{\gamma^*+\varepsilon}{\sigma} + G) + \beta(t_1)(\frac{\gamma^*+\varepsilon}{\sigma} + G)] \\
&\le \eta(t_1, y(t_1))[\frac{\gamma^*+\varepsilon}{\sigma}(\sigma + \alpha(t_1) + \beta(t_1)) + G(\alpha(t_1) + \beta(t_1)] \\
&\le -\eta(t_1, y(t_1))\sigma G \le 0.
\end{aligned} \tag{12}
$$

This contradicts the inequality in (11), and so (10) holds. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0$ and obtain (6).
(ii): By (6), one can know that the definition of h for (9) is reasonable. Denote

$$
H(\mu) = \mu + \alpha(t) + \beta(t)e^{h\mu\tau(t)}. \tag{13}
$$

For any fixed $t \ge t_0$, we see that

$$
H(0) = \alpha(t) + \beta(t) \le -\sigma < 0, \lim_{\mu\to+\infty} H(\mu) = +\infty \tag{14}
$$

and

$$
H'(\mu) = 1 + \tau(t)\beta(t)he^{\mu h\tau(t)} > 0. \tag{15}
$$

Therefore for any given $t \ge t_0$ there is a unique positive μ such that

$$
\mu + \alpha(t) + \beta(t)e^{h\mu\tau(t)} = 0, \tag{16}
$$

that means the (16) define an implicit function $\mu(t)$ for $t \ge t_0$. From that definition, one has $\mu^* \ge 0$. Next, we at first shall prove that for any positive constant ε,

$$
u(t) \le \frac{\gamma^*+\varepsilon}{\sigma} + Ge^{-\mu^* \int_{t_0}^{t} \eta(s,y(s))ds} \triangleq v(t), t \ge t_0. \tag{17}
$$

If (17) is not true, then there exists a constant $\xi > t_0$ such that

$$
u(\xi) = v(\xi), u'(\xi) > v'(\xi), u(t) \le v(t), t \in [t_0, \xi). \tag{18}
$$

Let $w(t) = v(t) - u(t)$, then we have

$$
\begin{aligned}
w'(\xi) =& v'(\xi) - u'(\xi) \\
\ge& - G\mu^*\eta(\xi, y(\xi))e^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds} - \eta(\xi, y(\xi))[\gamma(\xi) + \alpha(\xi)u(\xi) + \beta(\xi) \sup_{\xi-\tau(\xi)\le\theta\le\xi} u(\theta)] \\
>& - G\mu^*\eta(\xi, y(\xi))e^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds} - \eta(\xi, y(\xi))[\gamma^* + \varepsilon + \alpha(\xi)u(\xi) + \beta(\xi) \sup_{\xi-\tau(\xi)\le\theta\le\xi} u(\theta)].
\end{aligned} \tag{19}
$$

If $\xi - \tau(\xi) \geq t_0$, it follows from (19) that

$$\begin{aligned}
w'(\xi) \geq & - G\mu^* \eta(\xi, y(\xi)) e^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds} - \eta(\xi, y(\xi))[\gamma^* + \varepsilon + \alpha(\xi)(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}) \\
& + \beta(\xi)(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^* \int_{t_0}^{\xi-\tau(\xi)} \eta(s,y(s))ds})] \\
= & \eta(\xi, y(\xi))[-\frac{\gamma * + \varepsilon}{\sigma}(\sigma + \alpha(\xi) + \beta(\xi)) - Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}(\mu^* + \alpha(\xi) + \beta(\xi) e^{\mu^* \int_{\xi-\tau(\xi)}^{\xi} \eta(s,y(s))ds}).
\end{aligned} \tag{20}$$

From the define of function $\mu(t)$, we have

$$\begin{aligned}
& \mu^* + \alpha(\xi) + \beta(\xi) e^{\mu^* \int_{\xi-\tau(\xi)}^{\xi} \eta(s,y(s))ds} \\
& \quad = \mu^* + \alpha(\xi) + \beta(\xi) e^{\mu^* \int_{\xi-\tau(\xi)}^{\xi} \eta(s,y(s))ds} - \mu(\xi) - \alpha(\xi) - \beta(\xi) e^{h\mu(\xi)\tau(\xi)} \\
& \quad = (\mu^* - \mu(\xi)) + \beta(\xi)(e^{\mu^* \int_{\xi-\tau(\xi)}^{\xi} \eta(s,y(s))ds} - e^{h\mu(\xi)\tau(\xi)}) \leq 0.
\end{aligned} \tag{21}$$

Noting (5), therefore (20) yields

$$w'(\xi) = v'(\xi) - u'(\xi) \geq 0, \tag{22}$$

which contradicts the first inequality in (18).
If $\xi - \tau(\xi) < t_0$, it follows from (19) that

$$\begin{aligned}
w'(\xi) \geq & - G\mu^* \eta(\xi, y(\xi)) e^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds} - \eta(\xi, y(\xi))[\gamma^* + \varepsilon + \alpha(\xi)(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}) \\
& + \beta(\xi) \max\{\sup_{\theta \leq t_0} u(\theta), \sup_{t_0 \leq \theta \leq \xi} u(\theta)\}] \\
\geq & - G\mu^* \eta(\xi, y(\xi)) e^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds} - \eta(\xi, y(\xi))[\gamma^* + \varepsilon + \alpha(\xi)(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}) \\
& + \beta(\xi)(G + \frac{\gamma^* + \varepsilon}{\sigma})] \\
= & \eta(\xi, y(\xi))[-\frac{\gamma^* + \varepsilon}{\sigma}(\sigma + \alpha(\xi) + \beta(\xi)) - Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}(\mu^* + \alpha(\xi) + \beta(\xi) e^{\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}) \\
\geq & \eta(\xi, y(\xi))[-\frac{\gamma^* + \varepsilon}{\sigma}(\sigma + \alpha(\xi) + \beta(\xi)) - Ge^{-\mu^* \int_{t_0}^{\xi} \eta(s,y(s))ds}(\mu^* + \alpha(\xi) + \beta(\xi) e^{\mu^* \int_{\xi-\tau(\xi)}^{\xi} \eta(s,y(s))ds}).
\end{aligned} \tag{23}$$

Here we also obtain that (22) holds, which contradicts the first inequality in (18). Hence the inequality (17) must hold. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0$ and obtain (7). The proof of Theorem 2.1 is completed. □

Remark 2.2. Suppose that $\eta(t, y(t)) \equiv 1$ in Theorem 2.1, then we get Theorem 2.4 in [1].

3 Dissipativity of Lotka-Volterra functional differential equations

Definition 3.1. (See [1]) System (1) is said to be dissipative in H if there exists a bounded set $B \subset H$, such that for any given bounded set $\Phi \subset H$, there is a time $t^* = t^*(\Phi)$, such that for any given initial function $\varphi \in C_X[-\tau, 0]$ with φ contained in Φ for all $t \in [-\tau, 0]$, the values of the corresponding solution $y(t)$ of the problem are contained in B for all $t \geq t^*$. Here B is called an absorbing set of the problem.

Lemma 3.2. (Generalized Barbălat's lemma [2]) If
(H_1) $u : \mathbb{R}_+ \to \mathbb{R}^n$ is uniformly continuous;
(H_2) $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and $g(x) = 0$ iff $x = 0$;
(H_3) $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\mathcal{K}(\delta) \triangleq \inf\limits_{t\geq 0} \int_t^{t+\delta} h(s)ds > 0$, for any $\delta > 0$;
(H_4) $\lim_{t\to\infty} \int_0^t h(s)g(u(s))ds$ exists and is finite;
then $\lim\limits_{t\to\infty} u(t) = 0$.

Theorem 3.3. *Suppose that $y(t)$ is a solution of the problem (1) satisfying the condition (2), and there exists a constant $\sigma > 0$ such that*

$$\alpha(t) + \beta(t) \leq -\sigma < 0 \; for \, t \geq 0. \tag{24}$$

Then
(i)

$$\|y(t)\|^2 \leq \frac{\bar{\gamma}^*}{\sigma} + \bar{G}, t \geq 0. \tag{25}$$

(ii)

$$\|y(t)\|^2 \leq \frac{\bar{\gamma}^*}{\sigma} + \bar{G}e^{-\mu^* \int_{t_0}^t \eta(s,y(s))ds}, t \geq 0, \tag{26}$$

where $\bar{G} = \sup_{-\infty<\theta\leq 0} \|\varphi(\theta)\|^2$, $\bar{\gamma}^* = \sup_{0\leq t<+\infty} \gamma(t)$, *and* $\bar{\mu}^* \geq 0$ *is defined as*

$$\bar{\mu}^* = \inf_{t\geq 0}\{\mu(t) : \mu(t) + \alpha(t) + \beta(t)e^{h\mu(t)\tau(t)} = 0\}, \tag{27}$$

where

$$\bar{h} = \sup_{t\geq 0} \max_{(s,\|y\|^2)\in[t-\tau,t]\times[0,\frac{\bar{\gamma}^*}{\sigma}+\bar{G}]} \eta(t, y) < \infty. \tag{28}$$

Proof. To apply the result of Theorem 2.1, we have to extend the define of initial function in (1) as $y(t) = \varphi(-\tau)$ for $-\infty < t \leq \tau$.

Let

$$u(t) = \|y(t)\|^2 =< y(t), y(t) > . \tag{29}$$

From (2), we have

$$\begin{aligned} u'(t) &= \frac{d}{dt} < y(t), y(t) >= 2\Re < y(t), g(t, y(t), y(\cdot)) > \\ &\leq \eta(t, y(t))[\gamma(t) + \alpha(t)u(t) + \beta(t) \max_{t-\mu_2(t)\leq\theta\leq t-\mu_1(t)} u(\theta)] \\ &\leq \eta(t, y(t))[\gamma(t) + \alpha(t)u(t) + \beta(t) \max_{t-\mu_2(t)\leq\theta\leq t} u(\theta)]. \end{aligned} \tag{30}$$

Application of Theorem 2.1 to the above inequality yields (25) and (26). The proof is completed. $\square$

Corollary 3.4. *In addition to the conditions of Theorem 3.3 hold, further assume that* $\eta(s, y(s)) \geq \delta > 0$. *Then,*

(i) for any given $\varepsilon > 0$, *there exists a positive number* $t^*(\|\varphi\|_\infty, \varepsilon)$, *such that*

$$\|y(t)\|^2 \leq \frac{\bar{\gamma}^*}{\sigma} + \varepsilon, \forall t > t^*.$$

(ii) For any given $\varepsilon > 0$, *the problem (1) is dissipative with an absorbing set* $B = B(0, \sqrt{\frac{\bar{\gamma}^*}{\sigma} + \varepsilon})$.

Theorem 3.5: *In addition to the conditions of Theorem 3.3 hold, further assume that* $\eta(s, y(s)) = h(s)g(y(s))$, *where* g *and* h *satisfy* (H_2) *and* (H_3) *of Lemma 3.2, respectively. Then, for any given* $\varepsilon > 0$, *the problem (1) is dissipative with an absorbing set* $B = B(0, \sqrt{\frac{\bar{\gamma}^*}{\sigma} + \varepsilon})$.

Proof: We only need to consider the following two possible cases:

(i) If $\int_0^\infty \eta(s, y(s))\mathrm{d}s = \infty$, then from (26) we have $\lim_{t\to\infty} \|y(t)\| \leq \sqrt{\frac{\bar{\gamma}^*}{\sigma}}$.

(ii) If $\int_0^\infty \eta(s, y(s))\mathrm{d}s < \infty$, then $h(s)g(y(s)) \in L[0, \infty)$. From (25) and (30), we know that $\dot{y}(t)$ is bounded. So $y(t)$ is a uniformly continuous function. By Lemma 3.2, we have $\lim_{t\to\infty} y(t) = 0 \leq \sqrt{\frac{\bar{\gamma}^*}{\sigma}}$.

From above (i) and (ii), we know the problem (1) is dissipative with an absorbing set $B = B(0, \sqrt{\frac{\bar{\gamma}^*}{\sigma} + \varepsilon})$. The proof is completed. $\square$

Corollary 3.6. *In addition to the conditions of Theorem 3.3 hold. If* $\eta(s, y(s)) = g(y(s))$, *where* $g(\cdot)$ *is a continuous, positive definite function, then for any given* $\varepsilon > 0$, *system (1) is dissipative with an absorbing set* $B = B(0, \sqrt{\frac{\bar{\gamma}^*}{\sigma} + \varepsilon})$.

Remark 3.7. In the recent years, various generalized Halanay inequalities have been established and successfully applied to the problem of investigating the dissipativity of differential systems, [1,3-6]. However, the generalized Halanay inequalities in [1,3-6] are ineffective for studying the dissipativity of (1) due to the existence of the term "$\eta(t, y(t))$" of (1), unless one resorts to the rather restrictive condition that $\eta(t, y(t)) \geq \delta > 0$ (δ is a constant).

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