

Ostrowski type inequalities for functions whose derivatives are h -convex in absolute value

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Abstract

Some new inequalities of Ostrowski type for functions whose derivatives are h -convex in modulus are given. Applications for midpoint inequalities are provided as well.

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1 Introduction

1.1 Ostrowski Type Inequalities

Comparison between functions and integral means are incorporated in Ostrowski type inequalities as follows.

The first result in this direction is due to Ostrowski [38].

Theorem 1.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) with the property that $|f'(t)| \leq M$ for all $t \in (a, b)$. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M \quad (1.1)$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following results for absolutely continuous functions hold (see [29] – [31]).

Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous on $[a, b]$. Then, for all $x \in [a, b]$, we have:

$$\begin{aligned}
& \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty & \text{if } f' \in L_\infty[a, b]; \\ \frac{1}{(\alpha+1)^{\frac{1}{\alpha}}} \left[\left(\frac{x-a}{b-a} \right)^{\alpha+1} + \left(\frac{b-x}{b-a} \right)^{\alpha+1} \right]^{\frac{1}{\alpha}} \times (b-a)^{\frac{1}{\alpha}} \|f'\|_\beta & \text{if } f' \in L_\beta[a, b], \\ & \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ & \alpha > 1; \\ \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \|f'\|_1 & \end{cases}
\end{aligned} \tag{1.2}$$

where $\|\cdot\|_{[a,b],r}$ ($r \in [1, \infty]$) are the usual Lebesgue norms on $L_r[a, b]$, i.e., we recall that

$$\|g\|_{[a,b],\infty} := \operatorname{ess\,sup}_{t \in [a,b]} |g(t)|$$

and

$$\|g\|_{[a,b],r} := \left(\int_a^b |g(t)|^r dt \right)^{\frac{1}{r}}, \quad r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ and $\frac{1}{2}$ respectively are sharp in the sense presented in Theorem 1.1.

The above inequalities can also be obtained from the Fink result in [33] on choosing $n = 1$ and performing some appropriate computations.

If one drops the condition of absolute continuity and assumes that f is Hölder continuous, then one may state the result (see for instance [21] and the references therein for earlier contributions):

Theorem 1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be of r -Hölder type, i.e.,

$$|f(x) - f(y)| \leq H |x - y|^r, \quad \text{for all } x, y \in [a, b], \tag{1.3}$$

where $r \in (0, 1]$ and $H > 0$ are fixed. Then, for all $x \in [a, b]$, we have the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{H}{r+1} \left[\left(\frac{b-x}{b-a} \right)^{r+1} + \left(\frac{x-a}{b-a} \right)^{r+1} \right] (b-a)^r. \tag{1.4}$$

The constant $\frac{1}{r+1}$ is also sharp in the above sense.

Note that if $r = 1$, i.e., f is Lipschitz continuous, then we get the following version of Ostrowski's inequality for Lipschitzian functions (with L instead of H) (see for instance [13])

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) L, \tag{1.5}$$

where $x \in [a, b]$. Here the constant $\frac{1}{4}$ is also best.

Moreover, if one drops the condition of the continuity of the function, and assumes that it is of bounded variation, then the following result may be stated (see [15]).

Theorem 1.4. Assume that $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation and denote by $\bigvee_a^b(f)$ its total variation. Then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f) \quad (1.6)$$

for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

If we assume more about f , i.e., f is monotonically increasing, then the inequality (1.6) may be improved in the following manner [12] (see also the monograph [28]).

Theorem 1.5. Let $f : [a, b] \rightarrow \mathbb{R}$ be monotonic nondecreasing. Then for all $x \in [a, b]$, we have the inequality:

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{b-a} \left\{ [2x - (a+b)] f(x) + \int_a^b \operatorname{sgn}(t-x) f(t) dt \right\} \\ & \leq \frac{1}{b-a} \{ (x-a) [f(x) - f(a)] + (b-x) [f(b) - f(x)] \} \\ & \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [f(b) - f(a)]. \end{aligned} \quad (1.7)$$

All the inequalities in (1.7) are sharp and the constant $\frac{1}{2}$ is the best possible.

The case for the convex functions is as follows [18]:

Theorem 1.6. Let $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on $[a, b]$. Then for any $x \in (a, b)$ one has the inequality

$$\begin{aligned} & \frac{1}{2} \left[(b-x)^2 f'_+(x) - (x-a)^2 f'_-(x) \right] \\ & \leq \int_a^b f(t) dt - (b-a) f(x) \\ & \leq \frac{1}{2} \left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a) \right]. \end{aligned} \quad (1.8)$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $x = a$ or $x = b$.

For other Ostrowski's type inequalities for the Lebesgue integral, see [3]-[13] and [19].

Inequalities for the Riemann-Stieltjes integral may be found in [14], [16] while the generalization for isotonic functionals was provided in [17].

For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [20]

1.2 The Case of Derivatives that are Convex in Modulus

In [17], the author pointed out the following identity in representing an absolutely continuous function. Due to the fact that we use it throughout the paper we give here a short proof.

Lemma 1.7. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$, one has the equality:

$$f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b (x-t) \left(\int_0^1 f'[(1-\lambda)x + \lambda t] d\lambda \right) dt. \quad (1.9)$$

Proof. For any $t, x \in [a, b]$, $x \neq t$, one has

$$\frac{f(x) - f(t)}{x - t} = \frac{1}{x - t} \int_t^x f'(u) du = \int_0^1 f'[(1-\lambda)x + \lambda t] d\lambda,$$

showing that

$$f(x) = f(t) + (x-t) \int_0^1 f'[(1-\lambda)x + \lambda t] d\lambda \quad (1.10)$$

for any $t, x \in [a, b]$.

If we integrate (1.10) over t on $[a, b]$ and divide by $(b-a)$, we deduce the desired identity (1.9). Q.E.D.

Using the above lemma the following result can be pointed out improving Ostrowski's inequality [4].

Theorem 1.8. Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|$ is convex on (a, b) .

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) [|f'(x)| + \|f'\|_\infty]. \end{aligned} \quad (1.11)$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} [|f'(x)| + \|f'\|_p]. \end{aligned} \quad (1.12)$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

(iii) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{2} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] [(b-a) |f'(x)| + \|f'\|_1]. \end{aligned} \quad (1.13)$$

In order to extend this result for other classes of functions, we need the following preparatory section.

2 h -Convex Functions

2.1 Some Definitions

We recall here some concepts of convexities that are well known in the literature. Let I be an interval in \mathbb{R} .

Definition 2.1 ([32]). We say that $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y).$$

Some further properties of this class of functions can be found in [24], [25], [27], [37], [40] and [41]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2.2 ([27]). We say that a function $f : I \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions*, i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P -functions see [27] and [39] while for quasi convex functions, the reader can consult [26].

Definition 2.3 ([6]). Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense) or Breckner s -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [6], [7], [22], [23], [34], [35] and [43].

In order to unify the above concepts, S. Varošanec introduced the concept of h -convex functions as follows.

Assume that I and J are intervals in \mathbb{R} , $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I , respectively.

Definition 2.4 ([46]). Let $h : J \rightarrow [0, \infty)$ with h not identical to 0. We say that $f : I \rightarrow [0, \infty)$ is an h -convex function if for all $x, y \in I$ we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [46], [5], [36], [44], [42] and [45].

2.2 Inequalities of Hermite-Hadamard Type

In [42] the authors proved the following Hermite-Hadamard type inequality for integrable h -convex functions.

Theorem 2.5. Assume that $f : I \rightarrow [0, \infty)$ is an h -convex function, $h \in L[0, 1]$ and $f \in L[a, b]$ where $a, b \in I$ with $a < b$. Then

$$\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq [f(a) + f(b)] \int_0^1 h(t) dt. \quad (\text{HH})$$

If we write (HH) for $h(t) = t$, then we get the classical Hermite-Hadamard inequality for convex functions.

If we write it for the case of P -type functions, i.e., $h(t) = 1$, then we get the inequality

$$\frac{1}{2}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq f(a) + f(b), \quad (2.1)$$

provided $f \in L[a, b]$, that has been obtained in [27].

If f is integrable on $[a, b]$ and Breckner s -convex on $[a, b]$, for $s \in (0, 1)$, then by taking $h(t) = t^s$ in (HH) we get

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{s+1} \quad (2.2)$$

that was obtained in [22].

Since for the case of Godunova-Levin class of function we have $h(t) = \frac{1}{t}$, which is not Lebesgue integrable on $(0, 1)$, we cannot apply the left inequality in (HH).

We can introduce now another class of functions.

Definition 2.6. We say that the function $f : I \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1]$, if

$$f(tx + (1-t)y) \leq \frac{1}{t^s}f(x) + \frac{1}{(1-t)^s}f(y), \quad (2.3)$$

for all $t \in (0, 1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P -functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s -Godunova-Levin functions defined on I , then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We have the following Hermite-Hadamard type inequality.

Theorem 2.7. Assume that the function $f : I \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1)$. If $f \in L[a, b]$ where $a, b \in I$ and $a < b$, then

$$\frac{1}{2^{s+1}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{1-s}. \quad (2.4)$$

We notice that for $s = 1$ the first inequality in (2.4) still holds and was obtained for the first time in [27].

3 Inequalities for Functions Whose Derivatives are h -Convex in Modulus

3.1 The Case of $|f'|$ is h -Convex

The following result holds:

Theorem 3.1. Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|$ is h -convex on (a, b) with $h \in L[0, 1]$.

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) [|f'(x)| + \|f'\|_\infty] \int_0^1 h(t) dt. \end{aligned} \quad (3.1)$$

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} [|f'(x)| + \|f'\|_p] \int_0^1 h(t) dt. \end{aligned} \quad (3.2)$$

(iii) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times [(b-a) |f'(x)| + \|f'\|_1] \int_0^1 h(t) dt. \quad (3.3)$$

Proof. (i). Using (1.9) and taking the modulus, we have

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| &= \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f'[(1-\lambda)x + \lambda t] d\lambda dt \right| \\ &\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| |f'[(1-\lambda)x + \lambda t]| d\lambda dt := K \end{aligned}$$

Utilizing the h -convexity of $|f'|$ we have

$$\begin{aligned} K &\leq \frac{1}{b-a} \int_a^b \int_0^1 |x-t| [h(1-\lambda) |f'(x)| + h(\lambda) |f'(t)|] d\lambda dt \\ &= \frac{1}{b-a} \int_a^b |x-t| \left[|f'(x)| \int_0^1 h(1-\lambda) d\lambda + |f'(t)| \int_0^1 h(\lambda) d\lambda \right] dt \\ &= \frac{1}{b-a} \int_0^1 h(\lambda) d\lambda \int_a^b |x-t| [|f'(x)| + |f'(t)|] dt := M(x) \int_0^1 h(\lambda) d\lambda \\ &\leq \frac{1}{b-a} \int_0^1 h(\lambda) d\lambda \sup_{t \in [a,b]} [|f'(x)| + |f'(t)|] \int_a^b |x-t| dt \\ &= \left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] [|f'(x)| + \|f'\|_\infty] \int_0^1 h(\lambda) d\lambda \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) [|f'(x)| + \|f'\|_\infty] \int_0^1 h(\lambda) d\lambda, \end{aligned}$$

for any $x \in [a, b]$, and the inequality (3.1) is proved.

(ii). As above, we have

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \int_a^b |x-t| [|f'(x)| + |f'(t)|] dt := M(x) \int_0^1 h(\lambda) d\lambda.$$

Using Hölder's integral inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we get that

$$\begin{aligned} M(x) &\leq \frac{1}{b-a} \left(\int_a^b |x-t|^q dt \right)^{\frac{1}{q}} \left(\int_a^b (|f'(x)| + |f'(t)|)^p dt \right)^{\frac{1}{p}} \\ &= \frac{1}{b-a} \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1} \right]^{\frac{1}{q}} \| |f'(x)| + |f'| \|_p \end{aligned}$$

and the inequality (3.2) is proved.

(iii). We also have that

$$\begin{aligned}
 M(x) &\leq \sup_{t \in [a, b]} |x - t| \frac{1}{b - a} \int_a^b [|f'(x)| + |f'(t)|] dt \\
 &= \frac{1}{b - a} \max(x - a, b - x) \left[(b - a) |f'(x)| + \int_a^b |f'(t)| dt \right] \\
 &= \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] [(b - a) |f'(x)| + \|f'\|_1]
 \end{aligned}$$

and the inequality (3.3) is proved. Q.E.D.

The following particular case is interesting.

Corollary 3.2. With the assumptions of Theorem 3.1, we have the midpoint inequality

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 &\leq \frac{1}{4} (b-a) \left[\left| f'\left(\frac{a+b}{2}\right) \right| + \|f'\|_\infty \right] \int_0^1 h(t) dt,
 \end{aligned} \tag{3.4}$$

provided $f' \in L_\infty[a, b]$.

If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then, we have,

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 &\leq \frac{1}{2} (b-a)^{\frac{1}{q}} \left(\int_a^b \left[\left| f'\left(\frac{a+b}{2}\right) \right| + |f'(t)| \right]^p dt \right)^{\frac{1}{p}} \int_0^1 h(t) dt.
 \end{aligned} \tag{3.5}$$

If $f' \in L_1[a, b]$, then

$$\begin{aligned}
 &\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 &\leq \frac{1}{2} \left[(b-a) \left| f'\left(\frac{a+b}{2}\right) \right| + \int_a^b |f'(t)| dt \right] \int_0^1 h(t) dt.
 \end{aligned} \tag{3.6}$$

Remark 3.3. We observe that if $|f'|$ is convex on (a, b) , then Theorem 3.1 reduces to Theorem 1.8.

Assume that $|f'|$ is Breckner s -convex on $[a, b]$, for $s \in (0, 1)$.

(a) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{s+1} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) [|f'(x)| + \|f'\|_\infty]. \end{aligned} \quad (3.7)$$

(aa) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{(s+1)(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} [|f'(x)| + \|f'\|_p]. \end{aligned} \quad (3.8)$$

(aaa) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{s+1} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times [(b-a) |f'(x)| + \|f'\|_1]. \quad (3.9)$$

Assume that $|f'|$ is of s -Godunova-Levin type, with $s \in [0, 1)$.

(b) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{1-s} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) [|f'(x)| + \|f'\|_\infty]. \end{aligned} \quad (3.10)$$

(bb) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{1}{(1-s)(q+1)^{\frac{1}{q}}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{\frac{1}{q}} \times (b-a)^{\frac{1}{q}} [|f'(x)| + \|f'\|_p]. \end{aligned} \quad (3.11)$$

(bbb) If $f' \in L_1[a, b]$, then for any $x \in [a, b]$,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{1-s} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times [(b-a) |f'(x)| + \|f'\|_1]. \quad (3.12)$$

3.2 The Case of $|f'|^p$ is h -Convex

The following result also holds:

Theorem 3.4. Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on $[a, b]$ so that $|f'|^p$ with $p > 1$ is h -convex on (a, b) and $h \in L[0, 1]$.

(i) If $f' \in L_\infty[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \times [|f'(x)|^p + \|f'\|_\infty^p]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p}. \end{aligned} \quad (3.13)$$

(ii) If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{1/q} \times \left[(b-a) |f'(x)|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p}. \end{aligned} \quad (3.14)$$

(iii) If $f' \in L_p[a, b]$, then for any $x \in [a, b]$,

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times \left[|f'(x)|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p} \\ & \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \times \left[(b-a) |f'(x)|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p}. \end{aligned} \quad (3.15)$$

Proof. As in the proof of Theorem 3.1 we have

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| = \frac{1}{b-a} \left| \int_a^b \int_0^1 (x-t) f'[(1-\lambda)x + \lambda t] d\lambda dt \right| \\ & \leq \frac{1}{b-a} \int_a^b |x-t| \left(\int_0^1 |f'[(1-\lambda)x + \lambda t]|^p d\lambda \right)^{1/p} dt := K \end{aligned}$$

for any $x \in [a, b]$.

By Hölder's integral inequality we have

$$\begin{aligned} & \int_0^1 |f'[(1-\lambda)x + \lambda t]|^p d\lambda \leq \left(\int_0^1 1^q d\lambda \right)^{1/q} \left(\int_0^1 |f'[(1-\lambda)x + \lambda t]|^p d\lambda \right)^{1/p} \\ & = \left(\int_0^1 |f'[(1-\lambda)x + \lambda t]|^p d\lambda \right)^{1/p} \end{aligned}$$

for any $x \in [a, b]$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

Since $|f'|^p$ is h -convex on (a, b) with $h \in L[0, 1]$, then

$$\int_0^1 |f'|^p [(1-\lambda)x + \lambda t]^p d\lambda \leq [|f'(x)|^p + |f'(t)|^p] \int_0^1 h(\lambda) d\lambda,$$

for any $x \in [a, b]$.

Therefore

$$K \leq \frac{1}{b-a} \left(\int_0^1 h(\lambda) d\lambda \right)^{1/p} \int_a^b |x-t| [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \quad (3.16)$$

for any $x \in [a, b]$.

(i). Now, if $f' \in L_\infty[a, b]$ then

$$\begin{aligned} & \int_a^b |x-t| [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \\ & \leq \text{ess sup}_{t \in [a, b]} [|f'(x)|^p + |f'(t)|^p]^{1/p} \int_a^b |x-t| dt \\ & = [|f'(x)|^p + \|f'\|_\infty^p]^{1/p} \frac{1}{2} [(x-a)^2 + (b-x)^2] \end{aligned}$$

for any $x \in [a, b]$, and utilizing (3.16), the inequality (3.13) is proved.

(ii). If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality we have

$$\begin{aligned} & \int_a^b |x-t| [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \\ & \leq \left(\int_a^b |x-t|^q dt \right)^{1/q} \left(\int_a^b ([|f'(x)|^p + |f'(t)|^p]^{1/p})^p dt \right)^{1/p} \\ & = \left[\frac{(b-x)^{q+1} + (x-a)^{q+1}}{q+1} \right]^{1/q} [(b-a)|f'(x)|^p + \|f'\|_p^p]^{1/p} \\ & = \frac{(b-a)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \left[\left(\frac{b-x}{b-a} \right)^{q+1} + \left(\frac{x-a}{b-a} \right)^{q+1} \right]^{1/q} \times [(b-a)|f'(x)|^p + \|f'\|_1^p]^{1/p} \end{aligned}$$

for any $x \in [a, b]$, and by (3.16) we deduce the desired inequality (3.14).

(iii). If $f' \in L_p[a, b]$, then by Hölder's inequality we also have

$$\begin{aligned}
& \int_a^b |x - t| [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \\
& \leq \sup_{t \in [a, b]} |x - t| \int_a^b [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \\
& = \max\{x - a, b - x\} \int_a^b [|f'(x)|^p + |f'(t)|^p]^{1/p} dt \\
& = (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \| |f'(x)|^p + |f'|^p \|^{1/p} \\
& \leq (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \left(\int_a^b [|f'(x)|^p + |f'(t)|^p] dt \right)^{1/p} \\
& = (b - a) \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b - a} \right| \right] \left((b - a) |f'(x)|^p + \|f'\|_p^p \right)^{1/p}
\end{aligned}$$

for any $x \in [a, b]$.

Q.E.D.

The following midpoint type inequalities are of interest.

Corollary 3.5. With the assumptions of Theorem 3.4, we have the inequality

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{1}{4} (b-a) \left[\left| f'\left(\frac{a+b}{2}\right) \right|^p + \|f'\|_\infty^p \right]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p},
\end{aligned} \tag{3.17}$$

provided $f' \in L_\infty[a, b]$.

If $f' \in L_p[a, b]$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{1}{2(q+1)^{1/q}} (b-a)^{\frac{1}{q}} \times \left[(b-a) \left| f'\left(\frac{a+b}{2}\right) \right|^p + \|f'\|_p^p \right]^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p}.
\end{aligned} \tag{3.18}$$

If $f' \in L_p[a, b]$, then

$$\begin{aligned}
 & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{1}{2} \left\| \left| f'\left(\frac{a+b}{2}\right) \right|^p + |f'|^p \right\| \left(\int_0^1 h(t) dt \right)^{1/p} \\
 & \leq \frac{1}{2} \left((b-a) \left| f'\left(\frac{a+b}{2}\right) \right|^p + \|f'\|_p^p \right)^{1/p} \left(\int_0^1 h(t) dt \right)^{1/p}.
 \end{aligned} \tag{3.19}$$

Remark 3.6. The interested reader can state the corresponding particular inequalities for different h -convex functions. However the details are omitted.

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