QUERIES

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A Query on Sard's Theorem

Suppose F and G are one variable differentiable concave functions. Define the set A={a:F' and G' are differentiable at a} and the function

$$f(x,y) = \left(\frac{F'(x)}{xF'(x)+yG'(y)}, \frac{G'(y)}{xF'(x)+yG'(y)}\right)$$

Then $\text{Jac} f(a,b) = (aF'(a)+bG'(b))^{-3}[-F''(a)G'(b)^2 - F'(a)^2G''(b)]$ if $(a,b) \in A \times A$. Sard's (1958) theorem says that $f\{(a,b) \in A \times A; (\text{Jac f})(a,b) = 0\}$ is null. Since F and G are concave, if the Jacobian is zero at (a,b), then $F''(a) = 0 = G''(b)$. The next proposition can therefore be interpreted as a (small and particular) generalization of Sard's theorem valid at points of non-differentiability:

Proposition. *The set* $f\{(a,b) \in A \times A; \bar{D}F'(a) = 0 = \bar{D}G'(b)\}$ *is null.*

How far can one generalize Sard's theorem at points of non-differentiability? In particular a result that covers functions like $f(x,y) = \frac{\text{grad}U(x,y)}{(x,y)\cdot\text{grad}U(x,y)}$ would be useful in mathematical economics if U is quasi-concave.

References

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[2] A. Sard, *Images of critical sets*, Annals of Mathematics, **68** $n^{\underline{o}}2$ (1958).

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