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SOME ADDITIONAL NOTES ON GLOBALLY POROUS SETS*

Abstract

Two different notions of global porosity are discussed. Relations between them are discovered, and some new characterizations of globally porous sets are given. A connection to Dolženko's boundary value problem is emphasized. Finally the notion of a bilaterally globally porous set is defined and some properties of such sets are discused. Several errors in the literature are corrected.

1. An Introduction

A notion of global porosity appears for the first time (to my knowledge) in [3], where the definition and some basic properties were given. It was developed by the authors as a step in the solution of Dolženko's boundary value problem. The problem is as follows:

> Let f be a complex valued function, holomorphic in the open upper half plane I^+. We can define a set of singular points of f in this sense: a point $x \in \mathbb{R}$ is a singular point for f if there is a pair of open angles V, $V' \subset I^+$ with a common vertex x such that the cluster set of f at x with respect to V differs from that with respect to V'. Let $A_{VV}(f)$ be the set of all singular points of f. Dolženko's problem is to find the full characterization of such sets.

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Dolženko in [2] showed that all such sets are $G_{\delta\sigma}$ and σ-porous. He states the hypothesis that this is a full characterization. In [4], Kolesnikov presents the following characterization:

Theorem 1.1 (Kolesnikov) *Let $E \subseteq \mathbb{R}$. The following conditions are equivalent:*

(i) There is an arbitrary (real or complex valued) function f defined on I^+ such that $E = E_{VV}(f)$.

(ii) There is a function f holomorphic and bounded on I^+ such that $E = E_{VV}(f)$.

(iii) There is a sequence of closed sets $\{F_n\}$ with each $F_n \subseteq \mathbb{R}$ such that $E = \cup_{n=1}^{\infty} \mathrm{p}(F_n)$. (For the definition of $\mathrm{p}(F_n)$ see below – Definition 1.1.)

However he did not answer Dolženko's original question – whether or not these sets are exactly the σ-porous $G_{\delta\sigma}$ sets. T. Vessey gives in [5] the following results (using a notion of global porosity):

Lemma 1.2 *Let E be a $G_{\delta\sigma}$ subset of some globally porous set. Let $\alpha \in (0, \pi/2)$. Then there is a continuous function $f : I^+ \to [0,1]$ and $\beta \in (\alpha, \pi/2)$ such that*

(i) For each $x \in E$ the cluster set of f at x with respect to the angle $V_x(\alpha, \pi - \alpha)$ is equal to $[0, 1]$.

(ii) For each $x \in \overline{E}$ the cluster set of f at x with respect to $V_x(\beta, \pi - \beta)$ is equal to $\{0\}$.

(iii) For each $x \notin E$ and for each angle $V \subset I^+$ with vertex at x the cluster set of f at x with respect to V is equal to $\{0\}$.

Theorem 1.3 *Let E be a σ-globally porous $G_{\delta\sigma}$ set. Then there is a continuous function $f : I^+ \to [0,1]$ such that $E = E_{VV}(f)$.*

One can see that the notion of global porosity is a partial solution of Dolženko's problem. Acording to [3], chapter "Some negative results", it is not the complete solution. There is a set which is an $A_{VV}(f)$ for some f and which is not σ-globally porous. We will investigate this definition and correct an error in Corollary 3 of [3].

L. Zajíček gives in [7] a different definition of global porosity and states that both definitions are equivalent. However we will show that this too is in error; the second definition is more general than the first one. We characterize this second notion in several ways and investigate the relationship between the two definitions. We notice that the second definition is also a partial solution of Dolženko's problem in the same sense as the first one. Unfortunately again it is not the complete solution. Using arguments similar to those found in [3] it can be shown that the perfect porous set constructed in [3] is not σ-globally porous in the Zajíček sense. Finally we show that this second notion of global porosity is not a new notion at all; it is only a new name for something known before.

First we establish some basic notation. We will use the common notation of basic set theoretical notions, such as $\overline{M}$ – the closure of a set M, M^0 – interior of M, M' – the set of limit points of M and ∂M – the boundary of M. If r is a real number, we will use the following notation:

(i) $r + M = \{x = r + y : y \in M\}$ ($r - M$ is defined similarly).

(ii) $r \cdot M = \{x = ry : y \in M\}$.

By "connecting interval" of some closed set $E \subseteq \mathbb{R}$ we mean an arbitrary bounded component (possibly empty) of the complement of E. Thus when referring to a system $\{I_n\}_{n=1}^{\infty}$ of all pairwise disjoint connecting intervals of E we mean the I_n's can be empty for $n \geq n_0 \geq 1$.

A "portion" of a set E is an arbitrary nonempty intersection of E with an open interval.

As usual, the porosity of E at a point $x \in \mathbb{R}$ is defined by

$$\mathrm{p}(E,x) = \limsup_{h \to 0_+} \frac{\lambda\,(E,(x-h,x+h))}{2h}$$

where for an arbitrary interval I:

$$\lambda(E,I) = \sup\{|J| : J \text{ is an interval }, J \subseteq I \setminus E\},$$

"the length of the largest gap of E in the interval I". The right and left porosities (p_+ and p_-) are defined in the obvious way. Notice that some authors omit the coefficient $1/2$ in the definition of the porosity.

A set E is called porous at x (porous on right, left, bilaterally porous) if the porosity (right, left porosity, both right and left porosity) of E in the x is greater than 0.

TA set is called porous if it is porous at each of its points. A set is called uniformly porous if the porosity at each of its points is greater than some fixed $\varepsilon > 0$. The notions of bilaterally and uniformly bilaterally porous set are defined in the obvious way. A set is called σ-porous if it can be expressed as a countable union of porous sets.

Definition 1.1 *Let $E \subseteq \mathbb{R}$ and set* $\mathrm{p}(E) = \{x \in E' \cap E : \mathrm{p}(E, x) > 0\}$.

We will say that two intervals are concentric if they have the same geometry and have a common center. If I is an interval and $0 < r \in \mathbb{R}$ we define $r * I$ to be the interval concentric with I which has the same geometry as I and which is of length $r \cdot |I|$.

The binary operation $*$ is obviously monotone in both variables with respect to usual ordering of $\mathbb{R}$ and inclusion.

The following two lemmas are easy to prove.

Lemma 1.4 *Suppose $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Let $\mathcal{I}$ be the system of all connecting intervals of $\overline{E}$. Let $r > 0$ be such that for every $d > 0$ there exists an interval $I \in \mathcal{I}$, $|I| < d$ such that $x \in r * I$. Then* $\mathrm{p}(E, x) \geq \frac{1}{r+1}$.

PROOF. Fix x and take $I \in \mathcal{I}$ such that $x \in r * I$. The distance of x from the center of I is less than $r|I|/2$ and therefore $I \subseteq (x - h, x + h)$ with $h = (r + 1)|I|/2$. Thus $\lambda(E, (x + h, x - h)) \geq |I|$. Since the interval I can be arbitrarily small, $\mathrm{p}(E, x) \geq 1/(r + 1)$.

Lemma 1.5 *Let $E \subseteq \mathbb{R}$. Let $x \in (\inf(E), \sup(E))$, and let $\mathcal{I}$ denote the system of all connecting intervals of $\overline{E}$. Suppose there exists an $\varepsilon > 0$ and $r > 1$ such that $x \notin r * I$ for every interval $I \in \mathcal{I}$ with distance from x smaller than ε. Then* $\mathrm{p}(E, x) \leq \frac{1}{r+1}$.

PROOF. Take ε, r and x such that hypothesis is valid. Let $\mathrm{p}(E, x) > 1/(r+1)$. There exists arbitrarily small $h > 0$ such that $\lambda(E, (x-h, x+h)) > 2h/(r+1)$. Choose $h < \min(\varepsilon, x - \inf(E), \sup(E) - x)$. Let L be the longest gap of E in the interval $(x - h, x + h)$. Then, $|L| > \frac{2h}{r+1}$. The choice of h implies that $L = I \cap (x - h, x + h)$ for some $I \in \mathcal{I}$. As the distance of x from I is less than $h \leq \varepsilon$, it follows that $x \notin r * I$. The monotonicity of $*$ implies that $x \notin r * L$. Therefore the distance of x from the center of L is greater or equal than $r|L|/2$. Thus $h \geq (r + 1)|L|/2$ and $|L| \leq \frac{2h}{r+1}$, a contradiction.

2. Globally Porous Sets

Definition 2.1 *Let E be a bounded subset of $\mathbb{R}$, containing at least two elements. Take a system $\{I_n\}_{n=1}^{\infty}$ of all connecting intervals of $\overline{E}$. For each $N \in \mathbb{N}$ let $\mathrm{E}P(N)$ denote the set of all endpoints of intervals $I_1, \ldots, I_{N-1}$.*

If $r > 0$ exists such that for each $N \in \mathbb{N}$ there is a $M \in \mathbb{N}$, $M > N$ such that $E \setminus \mathrm{E}P(N) \subseteq \cup_{n=N}^{M} r * I_n$, then the set E is called r-globally porous. The empty set and singletons are defined to be r-globally porous for each $r > 0$.

A set is called globally porous if it is r-globally porous for some $r > 0$. A set is called σ-globally porous if it can be writen as a countable union of globally porous sets.

It is easy to see the definition does not depend on the ordering of the system of the connecting intervals of $\overline{E}$. Also it is clear that E is globally porous iff $\overline{E}$ is globally porous. As a result of this fact Humke and Vessey state that a set is σ-globally porous iff it is a subset of a σ-globally porous F_σ set ([3], Corollary 3). As we will show, only the necessity is true, namely:

Corollary 2.1 *Let E be a σ-globally porous set. There exists another σ-globally porous set F of type F_σ such that $E \subseteq F$.*

First we consider the porosity of globally porous sets at points of their closures. The following lemma can be easily proved using Lemma 1.4.

Lemma 2.2 *Let $E \subset \mathbb{R}$ be globally porous. There exists an $\varepsilon > 0$ such that for each $x \in \overline{E}$, $\mathrm{p}(E, x) > \varepsilon$.*[1]

Lemma 2.3 *Let E be a bounded subset of $\mathbb{R}$. Let I denote one of the components of the complement of $\overline{E}$ and let a denote the left (right) endpoint of I. If E is not porous on the right (left) at a then E is not globally porous.*

PROOF. For a contradiction consider E to be r-globally porous with some constant r. We may assume E is closed. It is easy to see that $F = E \cap [a, \infty)$ should be globally porous as well. So there exists an arbitrarily small connecting interval J of F such that $a \in r * J$. Thus F is porous on the right at a, which is a contradiction.

A globally porous set can contain a subset which is not globally porous.[2] It follows that globally porous sets can contain subsets which are only unilaterally porous at some point. The previous lemma describes the situations which cannot occur.

3. Divisions and Coverings

Definition 3.1 *By $\mathbb{R}^*$ we mean the the set of real numbers extended by elements $-\infty$ and $+\infty$. We will consider the usual topology and ordering of $\mathbb{R}^*$.*

In a sequel, the symbol $\mathbb{I}$ means a nonempty open interval $(a, b) \subseteq \mathbb{R}^*$.

Definition 3.2 *Two intervals intersecting each other in at most one point are said to be almost disjoint. A system of pairwise almost disjoint intervals is called an almost disjoint system of intervals.*

[1] *That is, $\overline{E}$ is uniformly porous.*

[2] It is easy to construct a globally porous set containing a sequence $\{1/n\}_{n=1}^{\infty}$.

Definition 3.3 *Let $a,\ b \in \mathbb{R}^*$, $a < b$ and let $\{a_n\}_{n=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers such that $\lim_{n\to-\infty} a_n = a$, and $\lim_{n\to\infty} a_n = b$. Then the system of intervals $D = \{[a_n, a_{n+1}] : n \in \mathbb{Z}\}$ will be called a division of $\mathbb{I} = (a,b)$. The norm of the division D, assigned $\nu(D)$, is defined as $\sup_{n\in\mathbb{Z}} \{a_{n+1} - a_n\}$.*

Let f be a strictly increasing function defined on the interval $[0,\infty)$ such that $f(0) = 0$. The division D of $\mathbb{I} = (a,b)$ is said to be controlled by the function f if both:

(a) If $a \in \mathbb{R}$, then $a_{n+1} - a_n \le f(a_n - a)$ holds for each $n \in \mathbb{Z}$.

(b) If $b \in \mathbb{R}$, then $a_{n+1} - a_n \le f(b - a_{n+1})$ holds for each $n \in \mathbb{Z}$.

Now suppose that $a, b \in \mathbb{R}$ and $a \le b$. A division of the closed *interval $[a,b]$ is defined as a finite almost disjoint system $D = \{I_n\}_{n=1}^{N}$ of closed intervals such that $\cup_{n=1}^{N} I_n = [a,b]$. If the interval $[a,b]$ is nondegenerate, the intervals of the division should be nondegenerate as well.*

The norm of the division D is $\max_{n=1,\dots,N}\{|I_n|\}$.

Definition 3.4 *A system $\mathcal{S}$ of intervals is called a (inner) covering of $\mathbb{I}$ if $\mathbb{I} = \cup\mathcal{S}$ and for each $I \in \mathcal{S}$, $\overline{I} \subseteq \mathbb{I}$. (We will omit the adjective "inner" in the sequel.)*

A covering $\mathcal{S}$ of $\mathbb{I}$ is called locally finite if for each $x \in \mathbb{I}$ there is a neighbourhood of x intersecting only finitely many intervals of $\mathcal{S}$.

In an obvious way an almost disjoint locally finite covering of $\mathbb{I}$ by closed intervals defines a division of $\mathbb{I}$.

Lemma 3.1 *Let $\mathcal{P}$ be a covering of $\mathbb{I}$ by open intervals such that for each $x \in \mathbb{I}$ $\overline{\bigcup_{I\in\mathcal{P},\, x\in I} I}$ is a compact subset of $\mathbb{I}$. Then $\mathcal{P}$ contains a locally finite covering $\mathcal{S}$ of $\mathbb{I}$.*

PROOF. Cover $\mathbb{I}$ by an almost disjoint system of compact intervals $\{K_i\}_{i=-\infty}^{\infty}$ such that for $x_i \in K_i$ we have $\lim_{i\to-\infty} x_i = a$ and $\lim_{i\to+\infty} x_i = b$. Let $\mathcal{P}_i = \{I \in \mathcal{P} : I \cap K_i \neq \emptyset\}$. It follows from the compactness of $\overline{\bigcup_{I\in\mathcal{P},\, x\in I} I}$ that for each i there are only finitely many j such that there is $I_j \in \mathcal{P}_i$ and $I_j \cap K_j \neq \emptyset$.

As $\mathcal{P}_i$ is a covering of K_i by open intervals, there exists a finite subcovering $\mathcal{Q}_i \subseteq \mathcal{P}_i$. Let $\mathcal{Q} = \cup_{i=1}^{\infty} \mathcal{Q}_i$. It is clear that $\mathcal{Q}$ is a covering of $\mathbb{I}$, so it remains to prove that $\mathcal{Q}$ is locally finite. First note that each $x \in \mathbb{I}$ has a neighborhood U_x which intersects at most two of the intervals K_i, $i \in \mathbb{Z}$. Assign their indices as $j = j(x)$ and $k = k(x)$ (with $j = k$ if there is only one such interval).

Since each interval $I \in \mathcal{Q}$ intersecting U_x intersects K_j or K_k, there are only finitely many systems $\mathcal{Q}_k$ which can contain such I. As all of them are finite, it follows that the covering $\mathcal{Q}$ is locally finite.

Now we state several elementary lemmas concerning locally finite coverings.

Lemma 3.2 *Let $\mathcal{S}$ be a system of open intervals such that for each $I \in \mathcal{S}$, $\overline{I} \subseteq \mathbb{I}$. Then, $\mathcal{S}$ is a locally finite covering of $\mathbb{I}$ iff for every compact $K \subseteq \mathbb{I}$ the system $\{I \in \mathcal{S} : I \cap K \neq \emptyset\}$ is nonempty and finite.*

Lemma 3.3 *Let $\mathcal{S}$ be a locally finite covering of $\mathbb{I}$. Then for each $I \in \mathcal{S}$ there are only finitely many $J \in \mathcal{S}$ such that $I \cap J \neq \emptyset$.*

Lemma 3.4 *Let $\mathcal{S}$ be a locally finite covering of $\mathbb{I}$, and let $\overline{\mathcal{S}} = \left\{\overline{I} : I \in \mathcal{S}\right\}$. Then $\overline{\mathcal{S}}$ is locally finite covering of $\mathbb{I}$ by closed intervals.*

The proof of the following technical lemma is trivial.

Lemma 3.5 *Let $I_1, \dots, I_k$ be intervals and suppose $k \geq 3$. If $\cap_{l=1}^{k} I_l \neq \emptyset$, then there are indices $i, j \leq k$ such that $\cup_{l=1}^{k} I_l = I_i \cup I_j$.*

Lemma 3.6 *Let $\mathcal{V}$ be a locally finite covering of $\mathbb{I}$. Then there is a covering $\mathcal{U} \subseteq \mathcal{V}$ of $\mathbb{I}$, such that each $I \in \mathcal{U}$ intersects at most two of the other intervals of $\mathcal{U}$.*

PROOF. We construct sequences $\mathcal{V}_p$ by induction.
1. Let $\mathcal{V}_1$ be an arbitrary sequence of all intervals $V_n^1 \in \mathcal{V}$. This is the first sequence.
2, Let $p \geq 1$. If $\mathcal{V}_p$ contains no triad of intervals I, J and K such that $I \cap J \cap K \neq \emptyset$, we are ready. Alternatively, if there is such a triad, we will construct a sequence $\mathcal{V}_{p+1}$. Let n_p be the first natural number for which there is a pair $i, j \in \mathbb{N}$, $i \neq n_p \neq j \neq i$, such that $V_{n_p}^p \cap V_i^p \cap V_j^p \neq \emptyset$. Using the previous lemma, we can see that one of these three intervals is contained in the union of the others. Assign its index as k. Clearly $k \geq n_p$. Now for each $n \in \mathbb{N}$ put

$$V_n^{p+1} = \begin{cases} V_n^p & \text{if } n < k, \\ V_{n+1}^p & \text{alternatively} \end{cases}$$

If the process does not stop after some step, we will obtain the sequences (coverings of $\mathbb{I}$) $\mathcal{V}_p \supset \mathcal{V}_{p+1}$, $p = 1, 2, \dots$ and the increasing (not necessarily strictly) sequence of natural numbers $n_1 \leq \dots \leq n_p \leq n_{p+1} \leq \dots$ such that for each $p \in \mathbb{N}$ each interval I_n^p for $n < n_p$ intersects only two of the other intervals from $\mathcal{V}_p$. The sequence $\{n_p\}$ is unbounded (Use Lemma 3.3 and the fact that all sequences $\mathcal{V}_p$ are locally finite coverings of $\mathbb{I}$). Put $\mathcal{U} = \cap_{p=1}^{\infty} \mathcal{V}_p$.

Let $I \in \mathcal{U}$. Then $I \in \mathcal{V}_p$ for each p and there is a p_I and $n < n_{p_I}$ such that $I = V_n^{p_I}$. Thus the cardinality of the system

$$\mathcal{S}_I^{p_I} = \{J \in \mathcal{V}_{p_I} : J \neq I,\ J \cap I \neq \emptyset\}$$

is at most 2. But the system $\mathcal{S}_I = \{J \in \mathcal{U} : J \neq I,\ J \cap I \neq \emptyset\}$ is contained in each of $\mathcal{S}_I^p$ including $\mathcal{S}_I^{p_I}$ and therefore its cardinality is at most 2.

It remains to show the $\mathcal{U}$ is a covering of $\mathbb{I}$. Take an arbitrary $x \in \mathbb{I}$. Since $\mathcal{V}$ is locally finite, the system $\mathcal{S}_x = \{I \in \mathcal{V} : x \in I\}$ is finite. If $\mathcal{S}_x \cap \mathcal{U} = \emptyset$ then there is a $p \in \mathbb{N}$ such that $\mathcal{V}_p \cap \mathcal{S}_x = \emptyset$, but $\mathcal{V}_p$ is a covering of $\mathbb{I}$, a contradiction.

4. II-globally Porous Sets

Definition 4.1 *A set $E \subseteq \mathbb{R}$ is called II-globally porous if there is a $c > 0$ such that for each $d > 0$ there is a division D of $\mathbb{R}$ with norm less or equal to d such that for each $I \in D$ $\lambda(E, I) > c \cdot |I|$.*

The following technical lemma will be useful in the sequel.

Lemma 4.1 *Let $E \subset \mathbb{R}$, and let I and J be closed intervals such that $I^0 \cap J^0 \neq \emptyset$, $I \setminus J \neq \emptyset$, $J \setminus I \neq \emptyset$. Suppose that $\lambda(E, I) > c_I|I|$ and $\lambda(E, J) > c_J|J|$ for some c_I, $c_J > 0$. Then there exists a pair of almost disjoint closed intervals $\hat{I} \subseteq I$, $\hat{J} \subseteq J$, $\hat{I} \cup \hat{J} = I \cup J$, such that $\lambda(E, \hat{I}) > (c_I/2)|\hat{I}|$ and $\lambda(E, \hat{J}) > (c_J/2)|\hat{J}|$.*

PROOF. Let $I = [a, b]$, $J = [c, d]$, $a < c < b < d$. Take two intervals K, L such that $K \subseteq I \setminus E$, $L \subseteq I \setminus E$, $|K| = \lambda(E, I)$, $|L| = \lambda(E, J)$. Assign components of $\mathbb{R} \setminus \overline{E}$ containing K and L as S and T, respectively. We will distinguish the five possible cases:

1. $S \neq T$ and $I \cap J \subseteq S \cup T$.

It follows either $I \cap J \subseteq S$ or $I \cap J \subseteq T$ because $S \cap T = \emptyset$ and the set $I \cap J$ is connected. If, for example, $I \cap J \subseteq S$, put $\hat{I} = I$, $\hat{J} = [b, d]$. Then $K \subseteq \hat{I}$, $L \subseteq \hat{J}$.

2. $S \neq T$ and $(I \cap J) \setminus (S \cup T) \neq \emptyset$.

In this case, there is a point $x \in I \cap J$ such that S is on the left of x and T is on the right of x. Put $\hat{I} = [a, x]$, $\hat{J} = [x, d]$. Again $K \subseteq \hat{I}$ and $L \subseteq \hat{J}$.

3. $S = T$ and the center of $S \cap [a, d]$ lies in $I \cap J$.

Let x denote this center and put $\hat{I} = [a, x]$, $\hat{J} = [x, d]$. Then,

$$\lambda(E, \hat{I}) \geq \frac{1}{2}\lambda(E, I) > \frac{c_I}{2}|I| \geq \frac{c_I}{2}|\hat{I}|.$$

The same is valid for $\hat{J}$.

4. $S = T$ and the center of $S \cap [a, d]$ lies to the right of $I \cap J$.
   In this case put $\hat{I} = I$ and $\hat{J} = [b, d]$. Then

$$\lambda(E, \hat{J}) \geq \frac{1}{2}\lambda(E, J) > \frac{c_J}{2}|J| \geq \frac{c_J}{2}|\hat{J}|.$$

5. $S = T$ and the center of $S \cap [a, d]$ lies to the left of $I \cap J$.
   Set $\hat{I} = [a, c]$ and $\hat{J} = J$. The same estimate we made for J in the previous case is now valid in this case for I.

Now we will introduce several characterizations of globally porous sets.

Theorem 4.2 *Let $E \subseteq \mathbb{R}$. The following conditions are equivalent:*

(i) E is II-globally porous.

(ii) There is a $c > 0$ such that for every $d > 0$, for an arbitrary interval $[a, b]$ and every $\varepsilon > 0$ there are $0 \leq \delta_1, \delta_2 \leq \varepsilon$ and a division D of the closed interval $[a - \delta_1, b + \delta_2]$ with norm less or equal to d such that $\lambda(E, I) > c|I|$ for each interval $I \in D$.

(iii) $\overline{E}$ is uniformly porous.

(iv) There is a $c > 0$ such that for arbitrarily small $d > 0$, for every $\mathbb{I} = (a, b)$ with $a,\ b \in \mathbb{R}^*$ and every continuous strictly monotone function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ there is a division D of $\mathbb{I}$ with norm less or equal to d (controlled by the function f) such that $\lambda(E, I) > c|I|$ for every interval $I \in D$.

Proof. a) (i) $\Rightarrow$ (ii): Given an interval $[a, b]$, $d > 0$ and $\varepsilon > 0$, put $d_0 = \min(d, \varepsilon)$. There is a division D of $\mathbb{R}$ with $\nu(D) < d_0$ such that D fulfills the definition of II-globally porous. If a (respectively b) is an endpoint of some interval of D, put $\delta_1 = 0$ ($\delta_2 = 0$). Otherwise there is an endpoint a_n of some interval of D in $(a - \varepsilon, a)$ (or, in the case of b, $(b, b + \varepsilon)$). In this case put $\delta_1 = a - a_n$ (respectively $\delta_2 = a_n - b$). The division D can now be restricted to the interval $[a - \delta_1, b + \delta_2]$.

b) (ii) $\Rightarrow$ (iii): Let $x \in \overline{E}$ and put $a < x < b$. Suppose $d = \varepsilon$ is arbitrary and small. There are $\delta_1, \delta_2 < \varepsilon$ and a division D of $[a - \delta_1, b + \delta_2]$ with norm less or equal to d such that for each $I \in D$, $\lambda(E, I) > c|I|$. This holds in particular for an interval $I \in D$ which contains x. (At least one such interval exists.) Thus x is contained in an arbitrary small interval I such that $\lambda(E, I) > c|I|$. It follows that $\mathrm{p}(E, x) > c/2$.

c) (iii) $\Rightarrow$ (iv): This is the most difficult implication. For $x \notin \overline{E}$, $\mathrm{p}(E,x) = 1$. Thus there is $\varepsilon > 0$ such that $\mathrm{p}(E,x) > \varepsilon$ for each $x \in \mathbb{I}$. For definiteness we suppose $a \in \mathbb{R}$ and $b = +\infty$.[3] Take a $d > 0$ and a strictly monotone continuous function $f : [0,\infty) \to [0,\infty)$ such that $f(0) = 0$. For every $x \in \mathbb{I}$ there is an open symmetric neighbourhood of x, $U_x \equiv (a_x, b_x)$ with length less than or equal to d such that $\lambda(E, U_x) > (\varepsilon/2)|U_x|$, $\overline{U_x} \subseteq \mathbb{I}$ and $|U_x| \le f(a_x - a)$. The system of all such neighbourhoods is a covering of $\mathbb{I}$.

Let $x \in \mathbb{I}$, and define $F_x = \overline{\cup_{x \in U_y} U_y}$. If $x - a > d$ then F_x is clearly a compact subset of $\mathbb{I}$, since $|U_y| < d$ for every y. A more complicated situation takes place if x is close to a. If $x - a \le d$, let $v = x - a$. Since the function f is strictly increasing and $f(0) = 0$, there exists $y_0 \in \mathbb{I}$ such that $f(y_0 - a) + y_0 - a \le v$. Then for each y such that $a_y \le y_0$ it follows that $|U_y| \le v - (y_0 - a) = x - y_0$ and thus $x \notin U_y$. But then, $(a, y_0) \cap F_x = \emptyset$ and therefore F_x is a compact subset of $\mathbb{I}$. The hypothesis of Lemma 3.1. is fulfilled, and hence we can choose from our covering a locally finite subcovering of $\mathbb{I}$, which we denote as $\mathcal{Q}$. Put $\mathcal{V} = \overline{\mathcal{Q}}$ Using Lemma 3.6. we choose from $\mathcal{V}$ a new subcovering $\mathcal{U}$ such that each $I \in \mathcal{U}$ intersects at most two of the other intervals of $\mathcal{U}$. The covering $\mathcal{U}$ of $\mathbb{I}$ is comprised of closed intervals and has these properties:

- For every $I \in \mathcal{U}$ is $\lambda(E, I) > (\varepsilon/2)|I|$.
- For every $I \in \mathcal{U}$ is $|I| < d$.
- For every $I \in \mathcal{U}$ is $|I| < f(s - a)$, where s is a center of the interval I.
- Every $I \in \mathcal{U}$ intersects at most two of the other intervals of $\mathcal{U}$.
- Every compact subset of $\mathbb{I}$ intersects only finitely many intervals of $\mathcal{U}$.

However, we need a covering which is also almost disjoint and construct such a covering by induction.

1) If $\mathcal{U}$ contains two intervals whose intersection is a singleton, define these two intervals as $\widehat{I_0}$ and $\widehat{J_0}$. Alternatively choose two arbitrary intervals $I, J \in \mathcal{U}$ which intersect each other and applying Lemma 4.1. take a new pair of intervals $\widehat{I_0}$ and $\widehat{J_0}$ such that

$$\begin{aligned} \widehat{I_0} \cup \widehat{J_0} &= I \cup J, \\ \widehat{I_0} \cap \widehat{J_0} & \quad \text{is a singleton} \end{aligned}$$

and

$$\begin{aligned} \lambda\left(E, \widehat{I_0}\right) &> \frac{\varepsilon}{4}\left|\widehat{I_0}\right|, \\ \lambda\left(E, \widehat{J_0}\right) &> \frac{\varepsilon}{4}\left|\widehat{J_0}\right|. \end{aligned}$$

[3]The other cases can be handled in a completely analogous manner

2) Let $n \geq 0$. Suppose $I_0, \ldots, I_{n-1}$, $J_0, \ldots, J_{n-1}$, $\widehat{I_n}$ and $\widehat{J_n}$ are intervals which are altogether pairwise almost disjoint, and such that these properties hold:

$$\begin{aligned}
\lambda(E, I_k) &> \frac{\varepsilon}{8}|I_k| \text{ for } k < n, \\
\lambda(E, J_k) &> \frac{\varepsilon}{8}|J_k| \text{ for } k < n, \\
\lambda\left(E, \widehat{I_n}\right) &> \frac{\varepsilon}{4}\left|\widehat{I_n}\right|, \\
\lambda\left(E, \widehat{J_n}\right) &> \frac{\varepsilon}{4}\left|\widehat{J_n}\right|, \text{ and}
\end{aligned}$$

each of them is contained in an interval belonging to $\mathcal{U}$. Let the interval in $\mathcal{U}$ containing I_k (J_k) be denoted by $\widetilde{I_k}$ ($\widetilde{J_k}$) for $k < n$, and that containing $\widehat{I_n}$ ($\widehat{J_n}$) be denoted as $\widetilde{I_n}$ ($\widetilde{J_n}$). Suppose too that

$$\cup_{k=0}^{n-1} I_k \cup \cup_{k=0}^{n-1} J_k \cup \widehat{I_n} \cup \widehat{J_n} = \cup_{k=0}^{n} \widetilde{I_k} \cup \cup_{k=0}^{n} \widetilde{J_k},$$

and $\widehat{I_n}$ and $\widehat{J_n}$ are only intervals intersecting some intervals of $\mathcal{U}$ different from $\widetilde{I_k}$, $\widetilde{J_k}$ for $k = 0, \ldots, n$.

We will construct intervals I_n, J_n and $\widehat{I_{n+1}}$, $\widehat{J_{n+1}}$. Let us show the construction of I_n and $\widehat{I_{n+1}}$. The sets J_n and $\widehat{J_{n+1}}$ will be constructed in a similar way.

There is only one interval $I \in \mathcal{U}$ which intersects $\widehat{I_n}$ and which is not among the intervals $\widetilde{I_k}$ or $\widetilde{J_k}$ for $k = 0, \ldots, n$. If this intersection is a singleton, put $I_n = \widehat{I_n}$ and $\widehat{I_{n+1}} = I$. Otherwise use Lemma 4.1., obtaining intervals I_n and $\widehat{I_{n+1}}$ such that $I_n \cup \widehat{I_{n+1}} = \widehat{I_n} \cup I$, $I_n \cup \widehat{I_{n+1}}$ is a singleton and

$$\begin{aligned}
\lambda(E, I_n) &> \frac{\varepsilon}{8}|I_n|, \\
\lambda(E, \widehat{I_{n+1}}) &> \frac{\varepsilon}{4}\left|\widehat{I_{n+1}}\right|.
\end{aligned}$$

The definitions of J_n and $\widehat{J_{n+1}}$ are made in a similar way.

In this manner we obtain an almost disjoint system of intervals I_k and J_k for $k = 0, \ldots$, which is a covering of $\mathbb{I}$. It is easy to see this system forms a division D of $\mathbb{I}$ for which $\lambda(E, I) > \frac{\varepsilon}{8}|I|$ for every $i \in D$.

d) (iv) $\Rightarrow$ (i): Choose $\mathbb{I} = \mathbb{R}$.

5. Relations between Both Definitions

To begin we state a simple corollary of the last theorem of the previous section.

Corollary 5.1 *Every globally porous set is II-globally porous.*

We now use induction to construct a II-globally porous set which is not σ-globally porous.

Put $F_1 = [0,1]$ and let $G_1 = \cup_{n=2}^{\infty}\left(\frac{1}{n}, \frac{1}{n} + \frac{1}{2n(n-1)}\right)$. Inductively put $F_n = F_{n-1} \setminus G_{n-1}$. Let $\mathcal{J}_n$ be the system of all components of $(F_n)^0$ and put $G_n = \cup_{I \in \mathcal{J}_n} G(I)$ where $G(I) = (\inf(I) + |I| \cdot G_1)$.[4]

At this point we have a sequence of closed sets $F_1 \supset F_2 \supset \dots$ and we let F be their intersection. Then F is a nonempty perfect set which we prove is II-globally porous. It is sufficient to find $\varepsilon > 0$ such that for each $x \in F$, $\mathrm{p}(F, x) \geq \varepsilon$.

Let $x \in F$. If $x \in \partial F_n$ for some n, then $\mathrm{p}(F, x) = 1/2$. Alternatively, suppose $x \in (F_n)^0$ for each n. Let $I_n = I_n(x)$ be the component of $(F_n)^0$ containing x. Since $x \in (F_{n+1})^0$, $x \in (I_n \setminus G(I_n))^0$. The intervals $J_n^k = \inf(I_n) + |I_n| \cdot [1/k, 1/(k+1)]$ for $k > 2$ cover I_n and x belongs to the right half of some J_n^k. The left half of this J_n^k does not intersect F.

For large n, $|I_n(x)|$ (and hence $|J_n^k|$) can be chosen to be arbitrarily small and hence, $\mathrm{p}(F, x) \geq 1/4$. Thus we can choose $\varepsilon = 1/4$.

Now we will show the F is not σ-globally porous. Suppose to the contrary that $F = \cup_{n=1}^{\infty} E_n$, where each E_n is globally porous. According to Baire's theorem there is an $n \in \mathbb{N}$ such that E_n is somewhere dense in F and hence $\overline{E_n}$ contains a portion of F. This portion contains some small copy, H, of F. From the construction, it is evident that $\inf(H)$ is the right endpoint of some component of $\mathbb{R} \setminus \overline{E_n}$. Using Lemma 1.4. together with the fact that E_n is globally porous iff $\overline{E_n}$ is, we conclude that E_n is not globally porous, a contradiction.

Theorem 5.2 *Let E be a bounded subset of $\mathbb{R}$. If E is II-globally porous then there is a globally porous set F such that $E \subseteq F$.*

PROOF. Let $\{(a_n, b_n)\}_{n=2}^{\infty}$ be the system of all pairwise disjoint connecting intervals of $\overline{E}$. Let $K_0 = (\inf(E) - 3, \inf(E))$, $K_1 = (\sup(E), \sup(E) + 3)$, $K_n = (a_n, b_n)$ for $n \geq 2$ and $S = \left\{\frac{1}{2^k}\right\}_{k=0}^{\infty}$. Put

$$F = E \ \cup \ \cup_{n=2}^{\infty}\left(a_n + \frac{|K_n|}{3} \cdot S\right) \cup \cup_{n=2}^{\infty}\left(b_n - \frac{|K_n|}{3} \cdot S\right) \cup$$

$$\cup(\sup(E) + 2 \cdot S) \cup (\inf(E) - 2 \cdot S)$$

[4] $G(I)$ is actually a small copy of G_1.

If D is a division of $\mathbb{R}$, $J \in D$ and $J \cap (\inf(E) - 3, \sup(E) + 3) \neq \emptyset$ choose $i(J)$ such that $|J \cap K_{i(J)}| = \lambda(E, J)$. Order all connecting intervals of $\overline{F}$ into a sequence $\{I_n\}_{n\in\mathbb{N}}$ such that $|I_n| \geq |I_{n+1}|$. For $N \in \mathbb{N}$ let $\mathrm{E}P(N)$ denote the set of all endpoints of intervals I_n for $n < N$. We want to find such $M > N$ that $F \setminus \mathrm{E}P(N) \subseteq \cup_{n=N}^{M} r * I_n$ for some fixed $r > 0$.

As E is II-globally porous, there is a $c > 0$ such that for every $d > 0$ there is a division D of $\mathbb{R}$ with norm less or equal than d such that $\lambda(E, J) > c|J|$ for each $J \in D$. For fixed $N \in \mathbb{N}$, put $s = |I_{N-1}|$ and $d = s/(2c)$. There is a division D of $\mathbb{R}$ with norm less or equal then d such that $\lambda(E, J) > c|J|$ for each $J \in D$.

Consider only those intervals of D intersecting $\overline{F} \setminus \mathrm{E}P(N)$. There are only finitely many such intervals, and we let m denote the number of them. Denote the intervals themselves as $\{J_j\}_{j=1}^{m}$.

Case 1. Suppose $J_j \cap \overline{E} \neq \emptyset$.

If the middle third of interval $K_{i(J_j)}$ is not among the intervals I_k for $k < N$, assign this third as $\hat{I}_j$. Clearly it is a connecting interval of $\overline{F}$. Otherwise, find a connecting interval of $\overline{F}$ in this way: from the definition of the index $i(J_j)$ it follows that $K_{i(J_j)} \cap J_j = Q_j$, where $|Q_j| > c|J_j|$. Thus there is an interval $L \subseteq Q_j \subseteq J_j$ with length $c|J_j|$ such that $\overline{L}$ contains an endpoint of $K_{i(J_j)}$. For definiteness suppose it is the right endpoint. Further we know that $|L| = c|J_j| < c\frac{s}{2c} = \frac{s}{2}$. Now let $s_j = \min\left\{|I_k| : k < N,\ I_k \subseteq K_{i(J_j)}\right\}$. Because $s_j \leq s$, $|L| < \frac{s_j}{2}$, and it follows that $L \subseteq \left[a_{i(J_j)}, a_{i(J_j)} + \frac{s_j}{2}\right]$. Put $\hat{I}_j = \left(a_{i(J_j)} + \frac{s_j}{2}, a_{i(J_j)} + s_j\right)$. Then $\hat{I}_j$ is a connecting interval of $\overline{F}$ lying in the set $K_{i(J_j)}$. As its length is $\frac{s_j}{2}$, it cannot be among the intervals I_k for $k < N$. For the left endpoint of $K_{i(J_j)}$ we obtain a similar situation, but the interval $\hat{I}_j$ is: $\hat{I}_j = \left(b_{i(J_j)} - s_j, b_{i(J_j)} - \frac{s_j}{2}\right)$.

Case 2. Suppose $J_j \cap \overline{E} = \emptyset$.

In this case, $J_j \subset K_{i(J)}$, and J_j is a closed interval containing no limit point of F. It is easy to see that among the connecting intervals I_k of $\overline{F}$ intersecting J_j there is the shortest one, I_m, with $m \geq N$. Put $\hat{I}_j = I_m$.

For each $j = 1, \dots, m$ there is $n_j \geq N$ such that $\hat{I}_j = I_{n_j}$. Put $M = \max\{n_j : j = 1, \dots, m\}$. Let $x \in F \setminus \mathrm{E}P(N)$. Then there exists a $j \in \{1, \dots, m\}$ such that $x \in J_j$.

If $J_j \cap \overline{E} = \emptyset$ (Case 2), then $\hat{I}_j$ is the shortest of connecting intervals of $\overline{F}$ intersecting J_j. Because of ordering of connecting intervals of $\overline{F}$, among the intervals $I_1, \dots, I_M$ are all connecting intervals of $\overline{F}$ which intersect J_j. Points of F belonging to the interval J_j are isolated points of F and thus it is easy to see that

$$(F \cap J_j) \setminus \mathrm{E}P(N) \subseteq \cup_{n=N}^{M} r * I_n$$

whenever $r > 1$. Suppose now $J_j \cap \overline{E} \neq \emptyset$ (Case 1). If $\hat{I}_j$ is the middle third of the interval $K_{i(J_j)}$, then $K_{i(J_j)} \subseteq 3 * \hat{I}_j$ and because $|K_{i(J_j)} \cap J_j| > c|J_j|$, it follows that $J_j \subseteq (6/c) * \hat{I}_j$. Otherwise we have an interval $L \subseteq K_{i(J_j)} \cap J_j$ with length $c|J_j|$ such that $L \subseteq 3 * \hat{I}_j$. But then $(6/c) * \hat{I}_j$ covers the entire interval J_j. Since $c \leq 1$, we have $6/c > 1$ and thus $F \setminus \mathrm{E}P(N) \subseteq \cup_{n=N}^{M} \frac{6}{c} * I_n$.

Corollary 5.3 *The σ-globally porous sets do not form a σ-ideal.*

Corollary 5.4 *The smallest σ-ideal containing all globally porous sets is the σ-ideal of σ-II-globally porous sets.*

One can see that Corollary 3. from [3] indeed does not hold. A corrected version is below.

Corollary 5.5 *Let $E \subseteq \mathbb{R}$. The following conditions are equivalent:*

(i) E is σ-II-globally porous.

(ii) There is a σ-globally porous set, F of type F_σ containing E.

(iii) There is a σ-II-globally porous set, F of type F_σ containing E.

Now we can reformulate Vessey's lemma from [5]:

Lemma 5.6 *Let $E \subseteq \mathbb{R}$ be II-globally porous and suppose $\alpha \in (0, \pi/2)$. Then there is a continuous function $f : I^+ \to [0,1]$ and $\beta \in (\alpha, \pi/2)$ such that*

(i) For each $x \in E$ the cluster set of f at x with respect to the angle $V_x(\alpha, \pi - \alpha)$ is equal to $[0,1]$.

(ii) For each $x \in \overline{E}$ the cluster set of f at x with respect to $V_x(\beta, \pi - \beta)$ is equal to $\{0\}$.

(iii) For each $x \notin E$ and for each angle $V \subset I^+$ with vertex at x, the cluster set of f at x with respect to V is equal to $\{0\}$.

As a result of this lemma we obtain the next theorem.

Theorem 5.7 *Let E be a σ-II-globally porous $G_{\delta\sigma}$ set. Then there is a continuous function $f : I^+ \to [0,1]$ such that $E = E_{VV}(f)$.*

By using Kolesnikov's characterization of A_{VV} sets and the next lemma it is possible to prove this theorem without Lemma 5.6. The next lemma is based on the Theorem 4.2. and an idea suggested to me by M. Zelený.

Lemma 5.8 *Let $E \subseteq \mathbb{R}$ be a II-globally porous G_δ set without isolated points. Then there is a closed set $H \subseteq \mathbb{R}$ such that $E = \mathrm{p}(H)$.*

The main idea is this: Express E as an intersection of a monotone sequence of open sets $\{G_n\}$. Then for each open set construct a system $\mathcal{J}_n$ of open intervals $K \subseteq G_n \setminus \overline{E}$ whose lengths are relatively small compared to their distances from the complement of G_n. On the other hand, these intervals should also determine sufficiently large "gaps of E" in order to assure the porosity of the final set. It is sufficient to make a division of I controlled by the function $f(x) = x^2$ for each component I of the G_n and simply take all "gaps" whose existence follows from the Theorem 4.2. item (iv).

The union of all such intervals is an open set G whose complement we denote as F. Adding some isolated points to the set F we obtain a closed set H such that $E = \mathrm{p}(H)$. The size of selected "gaps" assures porosity at points of E and the controlling function assures that there are no "new" points of porosity.

The complete proof is considerably more detailed.

6. Bilaterally Globally Porous Sets

Items (ii) and (iii) of Theorem 4.2. suggest two different ways to define a notion of a bilaterally globally porous set.

Definition 6.1 *Let set $E \subset \mathbb{R}$. The E is called bilaterally globally porous if there is a $c > 0$ such that for each $d > 0$ and each closed interval I there exists a division D of I with norm less than d such that for each $J \in D$, $\lambda(E, J) > c|J|$.*

Definition 6.2 *Let set $E \subset \mathbb{R}$. The E is called bilaterally globally porous if it has bilaterally uniformly porous closure.*

Theorem 6.1 *Definitions 6.1 and 6.2 are equivalent.*

PROOF. Suppose E satisfies the first definition with some constant c, and let $x \in E$. Put $I = [x, y]$ for some $y > x$, and set $d > 0$. Then, by definition 6.1., there is a division D of I with norm less than d such that for each $J \in D$, $\lambda(E, J) > c|J|$. There exists $J \in D$ such that $J = [x, x + h]$ and $h < d$. Thus $\lambda(E, (x, x + h)) > ch$. Since we can take an arbitrarily small d, we can obtain arbitrarily small h as well. Therefore $\mathrm{p}_+(E, x) \geq c$. In the same way we can obtain an estimate of $\mathrm{p}_-(E, x)$.

Now, on the other hand, suppose that E has bilaterally uniformly porous closure. Having an interval $I = [a, b]$ and $d > 0$, we need a division D of

I. We know that $\mathrm{p}_+(E,a) > \xi$ and $\mathrm{p}_-(E,b) > \xi$ for some $\xi > 0$, which does not depend on the interval I. Hence there are $0 < \varepsilon_a, \varepsilon_b < d$ such that $\lambda(E,[a,a+\varepsilon_a]) > \xi\varepsilon_a$ and likewise $\lambda(E,[b-\varepsilon_b,b]) > \xi\varepsilon_b$. Put $\varepsilon = \min(\varepsilon_a,\varepsilon_b)$ and let $a' = a+\varepsilon_a$, $b' = b-\varepsilon_b$.

By the Theorem 4.2. there exist $0 < \delta_a, \delta_b < \varepsilon$ and a division D of the interval $[a'-\delta_a, b'+\delta_b]$ with norm less than d such that for each $J \in D$, $\lambda(E,J) > c|J|$, where c is independent of a', b', d and ε.

The interval $[a'-\delta_a, b'+\delta_b]$ is a part of I. The point $a'-\delta_a$ belongs to the interval $J_a = [a, a+\varepsilon_a]$ and similarly, $b'+\delta_b \in J_b = [b-\varepsilon_b, b]$. Define a division D' of I in the following manner: if the interval $J \in D$ intersects each of J_a and J_b in at most one point, then $J \in D'$. If all intervals from D intersecting J_a in more than one point are part of J_a, then $J_a \in D'$. Otherwise there is only one interval $J \in D$ such that $J^0 \cap J_a^0 \neq \emptyset$ and $J \not\subset J_a$. By Lemma 4.1. there exists a pair of almost disjoint intervals $\hat{J}$ and $\hat{J}_a$ such that $\hat{J} \cup \hat{J}_a = J \cup J_a$, $\lambda(E,\hat{J}) > (c/2)|\hat{J}|$ and $\lambda(E,\hat{J}_a) > (\xi/2)|\hat{J}_a|$. Both $\hat{J}$ and $\hat{J}_a$ fall into D'. In the same way we manage the set J_b. We obtain the division D' of the interval I, whose norm is less than d and for each $J \in D'$, $\lambda(E,J) > \frac{\min(c,\xi)}{2}|J|$.

As for the other kinds of porosity, a set is called σ-bilaterally globally porous if it can be written as a countable union of bilaterally globally porous sets. The σ-bilaterally globally porous sets form a σ-ideal.

It is also apparent that each bilaterally globally porous set is II-globally porous. Moreover, each bounded bilaterally globally porous set is globally porous:

Theorem 6.2 *Suppose $E \subset \mathbb{R}$ is a bounded bilaterally globally porous set. Let $c > 0$ be the constant from definition 6.1. Then E is $(2/c)$-globally porous.*

PROOF. Put $a = \inf(E)$, $b = \sup(E)$, and let $\{I_n\}_{n=1}^{\infty}$ be a system of all connecting intervals of $\overline{E}$. Let $N \in \mathbb{N}$, $N \geq 2$. Then $(a,b)\setminus\cup_{n=1}^{N-1} I_n = \cup_{n=1}^{k} K_n$, where K_n are pairwise disjoint closed intervals (possibly degenerate). For each K_n and for every $d > 0$ there is a division D from the definition 6.1. For each $n = 1,\dots,k$ and for arbitrary d take such a division D_n of K_n.

Fix $x \in E \setminus \mathrm{E}P(N)$. Then $x \in K_n$ for some n and the interval K_n is nondegenerate. Hence, x belongs to an interval $J \in D_n$. Let K be the greatest component of $J \setminus \overline{E}$. Then $|K| = \lambda(E,J) > c|J|$, and therefore $J \subseteq (2/c) * K$. But $K \subseteq I_m$ for some $m \geq N$ and thus $J \subseteq (2/c) * I_m$. Since there are only finitely many of intervals K_n and each division D_n has only finitely many intervals, there is a $M > N$ such that $E \setminus \mathrm{E}P(N) \subseteq \cup_{n=N}^{M} \frac{2}{c} * I_n$. As N is arbitrary it follows that E is $(2/c)$-globally porous.

There is another relation between bilaterally globally porous sets and globally porous sets.

Theorem 6.3 *A set $E \subset \mathbb{R}$ is bilaterally globally porous iff there exists $r > 0$ such that every portion of E is r-globally porous.*

PROOF. The necessity immediately follows from the preceeding theorem.

For the sufficiency, take $r > 0$ such that every portion of E is r-globally porous. For $x \in \overline{E}$, let I be an arbitrary nondegenerate interval with left endpoint at x. The set $\overline{I \cap E}$ is r-globally porous and all its connecting intervals I_n are to the right of x. If x is the left endpoint of any I_n, then $\mathrm{p}_+(E,x) = 1$. Otherwise $x \notin \mathrm{E}P(N)$ for every N, and hence for every $N \in \mathbb{N}$ there is a $n > N$ such that $x \in r * I_n$. Since $|I_n| \to 0$ for $N \to \infty$, $\mathrm{p}_+(E,x) \geq 2/(r+1)$. In the same manner we can estimate the left porosity. Thus E is bilaterally globally porous.

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