Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 328–331

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SOME INTERESTING SMALL SUBCLASSES OF THE DARBOUX BAIRE 1 FUNCTIONS

Abstract

We show that in the space, $\mathcal{D}B_1$, of Darboux Baire 1 functions on the unit interval, equipped with the metric of uniform convergence, the subspace $\mathcal{G}$, consisting of those functions whose graph restricted to the points of continuity is dense in the entire graph, is closed and nowhere dense, and that the subspace, $\mathcal{D}B_1^*$, of Darboux Baire* 1 functions is first category relative to $\mathcal{G}$; or stated in an equivalent manner, the space of universally first return continuous functions is a closed nowhere dense subspace of the space of first return continuous functions, and that $\mathcal{D}B_1^*$ is first category in the space of universally first return continuous functions.

Recognizing that the adjectives "interesting" and "small" are subjective, we must confess that it is we who consider the classes to be discussed here interesting and relatively small. We shall specify which subclasses of $\mathcal{D}B_1$, the collection of real-valued, Darboux, Baire 1 functions on $[0,1]$. we are referring to, indicate briefly why we have found them interesting, and show in what sense we consider them small.

The collection $\mathcal{D}B_1$ plays a significant role in real analysis and has been studied extensively, with numerous known characterizations. For example, one striking characterization states that $f \in \mathcal{D}B_1$ if and only if there is a homeomorphism h of $[0,1]$ onto itself for which $f \circ h$ is approximately continuous

Key Words: Baire 1, Baire* 1, Darboux, first return

Mathematical Reviews subject classification: Primary 26A21; Secondary 26A24

Received by the editors July 17,1993, Revised August 24, 1993

([6], Theorem 2). Recently, the present authors [2] have shown that $\mathcal{D}B_1$ is the collection of all *first return continuous functions.*

We pause here to explain first return continuity by reviewing some terminology from [2] and [3]. By a *trajectory* we mean any sequence $\{x_n\}_{n=0}^{\infty}$ of distinct points in $(0,1)$, which is dense in $[0,1]$. One method of specifying a trajectory is to assign an enumeration or *ordering* to a given countable dense subset D of $(0,1)$. We refer to such a set D as a *support* set. Let $\{x_n\}$ be a fixed trajectory. For a given interval $(a,b) \subset [0,1]$, $r(a,b)$ will be the first element of the trajectory in (a,b). For $0 \leq y < 1$, the *right first return path to* y, R_y^+, is defined recursively via

$$y_1^+ = 1, \text{ and } y_{k+1}^+ = r(y, y_k^+).$$

For $0 < y \leq 1$, the *left first return path to* y, R_y^-, is defined similarly. For $0 < y < 1$, we set $R_y = R_y^+ \cup R_y^- \cup \{y\}$, and $R_0 = \{0\} \cup R_0^+$, $R_1 = R_1^- \cup \{1\}$. The collection $\mathcal{R} \equiv \{R_y : y \in [0,1]\}$ forms a *path system*, which we refer to as the *first return path system determined by the trajectory* $\{x_n\}$. Let $f : [0,1] \to \mathbb{R}$. We say that f is *first return continuous on* $[0,1]$ *with respect to a trajectory* $\{x_n\}$ provided that at each $y \in [0,1]$ we have

$$\lim_{\substack{t \to y \\ t \in R_y \setminus \{y\}}} f(t) = f(y).$$

The function f is said to be *first return continuous on* $[0,1]$ if there exists a trajectory $\{x_n\}$ such that f is first return continuous on $[0,1]$ with respect to $\{x_n\}$.

Next, we let $\mathcal{G}$ denote the subclass of $\mathcal{D}B_1$ consisting of those functions for which the graph of the restriction of f to $C(f)$, the points of continuity of f, is dense in the graph of f; *i.e.*, $f \in \mathcal{G}$ if and only if $f \in \mathcal{D}B_1$ and $f|C(f)$ is dense in f. Paralleling the characterizations of $\mathcal{D}B_1$ mentioned above, we note two characterizations of $\mathcal{G}$: In [3] it was shown that $f \in \mathcal{G}$ if and only if there exists a homeomorphism h of $[0,1]$ onto itself for which $f \circ h$ is an approximately continuous function which is continuous almost everywhere; furthermore, $\mathcal{G}$ coincides with the class of *universally first return continuous functions.* (A function f is said to be *universally first return continuous on* $[0,1]$, if for every support set D there is an ordering $\{x_n\}$ of D, such that f is first return continuous on $[0,1]$ with respect to $\{x_n\}$ [3].)

The third, even smaller, class that we mention is the class $\mathcal{D}B_1^*$ of Darboux functions belonging to $\mathcal{B}_1^*$, the class Baire* 1. (A function f belongs to $\mathcal{B}_1^*$ if each perfect set contains a portion on which the restriction of f is continuous.) Again, this class has proved to be valuable in the study of real analysis; for example, it is known to contain the approximately differentiable functions. Furthermore, O'Malley [5] showed that $\mathcal{DB}_1^* \subset \mathcal{G}$. Thus we have that the Darboux, Baire* 1 functions are contained in the universally first return continuous functions (the functions in $\mathcal{G}$), which are, in turn contained in the class of first return continuous functions (the Darboux, Baire 1 functions). Equipping $\mathcal{DB}_1$ with the metric

$$d(f,g) = \min\{1, \sup(|f(x) - g(x)|)\}$$

of uniform convergence, we wish to observe that topologically $\mathcal{G}$ is small in $\mathcal{DB}_1$, and $\mathcal{DB}_1^*$ is small in $\mathcal{G}$. More specifically, we show the following.

Theorem 1 *The space $\mathcal{G}$ is closed and nowhere dense in $\mathcal{DB}_1$; furthermore, $\mathcal{DB}_1^*$ is of first category relative to $\mathcal{G}$.*

PROOF. Let us first prove that $\mathcal{G}$ is closed and nowhere dense in $\mathcal{DB}_1$. That $\mathcal{G}$ is closed is straightforward. To show that $\mathcal{G}$ is nowhere dense in $\mathcal{DB}_1$, we will construct a function $g \in \mathcal{DB}_1 \setminus \mathcal{G}$ arbitrarily close to a given function $f \in \mathcal{G}$. Let $f \in \mathcal{G}$ and $\epsilon > 0$. By the Maximoff-Preiss theorem [6], there exists a homeomorphism $h : I \to I$ such that $f \circ h$ is approximately continuous. Let (a, b) be an interval of length less than $\frac{\epsilon}{2}$ such that $f(h(I)) \cap (a, b) \neq \emptyset$. Since $f(h)$ is approximately continuous, each of $(f(h))^{-1}((-\infty, a])$ and $(f(h))^{-1}([b, \infty))$ is G_δ and density-closed. Let $G_0 = (f(h))^{-1}((-\infty, a])$, $G_1 = (f(h))^{-1}([b, \infty))$ and $M = I \setminus (G_0 \cup G_1)$. Using repeated applications of the Lusin-Menchoff theorem (Theorem 6.4, p.27, [1]), we may obtain a G_δ and density-closed set $H \subset M$ such that H and $M \setminus H$ are dense in M. Now by Zahorski's theorem (Theorem 6.5, p.28, [1]), we obtain an approximately continuous function $u : I \to \mathbb{R}$ such that

1. $u(G_0 \cup H) = a$,
2. $u(G_1) = b$, and
3. $a < u(x) < b$ for all $x \in M \setminus H$.

Let $v : I \to \mathbb{R}$ be such that $v(x) = f(h(x))$ if $x \notin M$, otherwise let $v(x) = u(x)$. Note that v is an approximately continuous function which is within ϵ of $f(h)$. Also $v(I) \cap (a, b) \neq \emptyset$, but no point of continuity of v maps into (a, b). Now let $g = v(h^{-1})$. Then, g is within $\frac{\epsilon}{2}$ of f, $g(I) \cap (a, b) \neq \emptyset$, and no point of continuity of g maps into (a, b). Therefore $g \notin \mathcal{G}$, which completes the proof of the first part.

Let us now show that $\mathcal{DB}_1^*$ is first category in $\mathcal{G}$. Let $\{[a_n, b_n]\}$ be an enumeration of intervals with rational endpoints which are contained in $(0, 1)$. For each positive integer n, let $F_n = \{f \in \mathcal{G}\colon f \text{ is continuous on } [a_n, b_n]\}$. Note that F_n is a closed subset of $\mathcal{G}$. To show that F_n is nowhere dense in $\mathcal{G}$, we will construct a function $g \in \mathcal{G} \setminus \mathcal{D}B_1^*$ which is arbitrarily close to a given function $f \in F_n$. Let $\epsilon > 0$ and $f \in F_n$. Let $\{r_i\}$ be an enumeration of the rationals in (a_n, b_n). For each positive integer i, let u_i be a continuous function such that

1. $u_i(r_i) = 0$,
2. u_i is zero on $[0,1] \setminus [a_n, b_n]$, and
3. $u_i(x) = \sin(\frac{1}{x-r_i})$ on $J_i \setminus \{r_i\}$ where J_i is an open interval containing r_i.

Let

$$g(x) = f(x) + \frac{\epsilon}{2} \cdot \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot u_i(x).$$

Then g is within ϵ of f and $g \in \mathcal{G}$; however, $g \notin \mathcal{B}_1^*$. Therefore F_n is nowhere dense in $\mathcal{G}$. Since $\mathcal{D}B_1^* \subset \bigcup_{n=1}^{\infty} F_n$, we have that $\mathcal{D}B_1^*$ is of first category relative to $\mathcal{G}$, completing the proof.

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