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THE S–HENSTOCK INTEGRATION AND THE APPROXIMATELY STRONG LUSIN CONDITION

1.

One approach to generalized integrals is to make use of Lusin's condition N [1, 2]. P. Y. Lee [4] introduced a concept which lies between absolute continuity and Lusin's condition N, and called it the strong Lusin condition. Recently, using the strong Lusin condition, Lee and Vyborny [5] defined the SL-integral. Moreover they showed the equivalence of the KH and SL-integrals. R. Gordon [3] gave an extension of the Henstock integral to the S-Henstock integral. Liao and Chew [6] extended the strong Lusin condition to the approximately strong Lusin condition. In this note, using the approximately strong Lusin condition we define the ASL integral. In addition we show that the ASL and SH integrals are equivalent.

2.

We present some notation and defintions to be used later.

Throughout this note we will consider real valued functions defined on a closed interval $[a,b]$. Let $P = \{([a_i,b_i],x_i) : i = 1,2,\ldots,n\}$ denote a finite collection of nonoverlapping tagged intervals in $[a,b]$. We call P a partial division of E, if $x_i \in E,\ E \subset [a,b]$.

Definition 1 (Gordon). *A distribution S on $[a,b]$ is a collection of measurable sets $\{S_x : x \in [a,b]\}$ in $[a,b]$ such that $x \in S_x$ and x is a point of density of S_x. For each $x \in [a,b]$ let $I_x = \{[c,d] : x \in [c,d]$ and $c,d \in S_x\}$ let δ be a positive function defined on $[a,b]$. A collection P of tagged intervals is S-subordinate to δ if $d - c < \delta(x)$ and $[c,d] \in I_x$, whenever $([c,d],x) \in P$.*

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Definition 2 *(Liao and Chew). Given $F : [a,b] \to \mathbb{R}$, we say that F satisfies the approximately strong Lusin condition, or ASL, on a set $H \subset [a,b]$ if and only if there exists a distribution S on H such that for every set E of measure zero and $\varepsilon > 0$ there exists a positive function γ on H such that*

$$\sum_{i=1}^{n} |F(v_i) - F(u)_i)| < \varepsilon,$$

whenever $P = \{([u_i, v_i], x_i);\ i = 1, 2, \ldots, n\}$ on $E \cap H$ is S-subordinate to γ.

3.

In this section we will define the ASL-integral and show the equivalence of the ASL and SH integrals.

Lemma 1 *If a function F satisfies ASL on $[a,b]$, then F satisfies the Lusin condition N.*

PROOF. Suppose to the contrary that F doesn't satisfy Lusin's condition N. There is a measurable set $E_0 \subset [a,b]$ such that $mE_0 = 0$ but $m^*F(E_0) > 0$ where m^* denotes outer Lebesgue measure. First show that $m^*F(E_0)$ is finite. By applying the definition of ASL on $[a,b]$, there exists a distribution S on $[a,b]$. Let $\varepsilon = \frac{1}{4}m^*F(E_0)$. Choose a positive function γ on $[a,b]$ such that $\sum_{i=1}^{n} |F(v_i) - F(u_i)| < \varepsilon$, whenever P on E_0 is $S-$ subordinate to γ. For each $x \in [a,b]$ choose $\delta_1(x) > 0$ so that $\frac{m(S_x \cap I)}{mI} > \frac{2}{3}$, whenever I is an interval containing x with $mI < \delta_1(x)$. This defines a positive function δ_1 on $[a,b]$. Let $\delta(x) = \min\{\gamma(x), \delta_1(x)\}$. Then δ is a positive function on E_0. For each positive integral n, let $E_n = \{x \in E_0 : \delta(x) > \frac{1}{n}\}$. Then $E_0 = \cup_{n=1}^{\infty} E_n$ and $\lim_{n\to\infty} m^*F(E_n) = m^*F(E_0)$, (cf, [8, p. 213]). There is a positive integer N such that

$$m^*F(E_N) > \frac{2}{3}m^*F(E_0). \tag{3.1}$$

Let $x_0 = \inf\{x : x \in E_N\}$ and $x_s = \sup\{x : x \in E_N\}$. Choose a positive integer M such that $\frac{b-a}{M} < \frac{1}{N}$ and $\frac{b-a}{M-1} \geq \frac{1}{N}$. Let $x_i = x_0 + \frac{x_s - x_0}{M}$, $i = 1, 2, \ldots, M$. Consequently

$$m^*F(E_N) \leq \sum_{i=1}^{M} m^*F(E_n \cap [x_{i-1}, x_i]). \tag{3.2}$$

We claim that for each i with $m^*F(E_N \cap [x_{i-1}, x_i]) > 0$ there exist $a_i, b_i \in E_N \cap [x_{i-1}, x_i]$ such that

$$|F(b_i) - F(a_i)| > \frac{1}{2}m^*F(E_N \cap [x_{i-1}, x_i]) - \frac{\theta}{N}, \tag{3.3}$$

where $0 < \theta < \frac{m^*F(E_0)}{12(b-a+1)}$. Suppose not. Then for all $x, y \in E_N \cap [x_{i-1}, x_i]$, we have $|F(x) - F(y)| \le \frac{1}{2}m^*F(E_N \cap [x_{i-1}, x_i]) - \frac{\theta}{N}$. We fix one variable x and denote it by a_i. Then $|F(a_i) - f(y)| \le \frac{1}{2}m^*F(E_N \cap [x_{i-1}, x_i]) - \frac{\theta}{N}$ for all $y \in E_N \cap [x_{i-1}, x_i]$. This implies

$$F(E_N \cap [x_{i-1}, x_i]) \subset$$

$$\left[F(a_i) - \frac{1}{2}m^*F(E_N \cap [x_{i-1}, x_i]) + \frac{\delta}{N}, F(a_i) + \frac{1}{2}m^*F(E_N \cap [x_{i-1}, x_i]) - \frac{\delta}{N}\right].$$

Consequently, $m^*F(E_N \cap [x_{i-1}, x_i]) \le m^*F(E_N \cap [x_{i-1}, x_i]) - \frac{2\delta}{N}$, a contradiction. Hence (3.3) holds.

Note that $\delta(a_i) > \frac{1}{N}$ and $\delta(b_i) > \frac{1}{N}$, so that $\delta_1(a_i) > \frac{1}{N}$ and $\delta_1(b_i) > \frac{1}{N}$. This implies

$$\frac{m([a_i, b_i] \cap S_{a_i})}{m([a_i, b_i])} > \frac{2}{3}, \quad \frac{m([a_i, b_i] \cap S_{b_i})}{m([a_i, b_i])} > \frac{2}{3}. \tag{3.4}$$

Therefore there is a $c_i \in S_{a_i} \cap S_{b_i} \cap [a_i, b_i]$. Let $P = \{([a_i, c_i], a_i) \cup ([c_i, b_i], b_i) : i = 1, 2, \ldots, M\}$. Then the partial division P on E_0 is S-subordinate to γ. Thus we have

$$\sum_{i=1}^{M} |F(b_i) - F(c_i)| + \sum_{i=1}^{M} |F(c_i) - F(a_i)| < \frac{1}{4}m^*F(E_0). \tag{3.5}$$

Clearly from (3.5) we get

$$\sum_{i=1}^{M} |F(b_i) - F(a_i)| > \frac{1}{4}m^*F(E_0). \tag{3.6}$$

According to (3.1), (3.2) and (3.3) this yields

$$\sum_{i=1}^{M} |F(b_i) - F(a_i)| > \frac{1}{3}m^*F(E_0) - \frac{M}{N}\delta. \tag{3.7}$$

Combining (3.6) and (3.7) gives $\frac{1}{4}m^*F(E_0) > \frac{1}{3}m^*F(E_0) - \frac{M}{N}\delta$. Note that $\frac{M}{N}\theta < (b-a+1)\theta$ and $\theta < \frac{m^*F(E_0)}{12(b-a+1)}$. This is a contradiction.

If $m^*F(E_0) = \infty$, then $\lim_{i\to\infty} m^*(F(E_0) \cap [-i, i]) = m^*F(E_0)$. We can find a subset E_1 of E_0 such that $m^*F(E_1) > 0$ and $mE_1 = 0$. Therefore, we have reduced the problem to the above case. □

Remark. This lemma answers an open problem which was asked by Liao and Chew [6]: Does ASL imply Lusin condition N?

We shall use following theorem.

Theorem 1 *(O'Malley). Let f be a function defined on $[a, b]$ and*

(1) *f is Baire class 1,*

(2) *$ap \limsup_{x \to x_0^-} f(x) \leq f(x_0) \leq ap \limsup)_{x \to x_0^+} f(x)$ for every x_0, and*

(3) *interior $[f(\{x : f_{ap}^+(x) \leq 0\})] = \emptyset$.*

Then f is nondecreasing.

Lemma 2 *Suppose F satisfies ASL and $F'_{ap}(x) \geq 0$. Then F is nondecreasing.*

PROOF. Let $G(x) = F(x) + \varepsilon x$ where $\varepsilon > 0$. It is clear that the sum of two ASL functions is an ASL function. This shows that G satisfies ASL. By Lemma 1, G satisfies Lusin's condition N. This implies that $mG\{x : G_{ap}^+(x) \leq 0\} = 0$. Since an ASL function is approximately continuous,it is clear that G satisfies the three conditions of O'Malley's Theorem 1. We conclude that G is nondecreasing. Let $\varepsilon \downarrow 0$. We get that F is nondecreasing. □

A routine argument preves the following theorem.

Theorem 2 *If a function F satisfies ASL and $F'_{ap}(x) = 0$ a.e. on $[a, b]$, then F is constant on $[a, b]$.*

Definition 3 *A function f is said to be ASL-integrable on $[a, b]$ if there exists an ASL the function F on $[a, b]$ such that $F'_{ap}(x) = f(x)$ almost everywhere on $[a, b]$. We define the ASL-integral of f on $[a, b]$ by*

$$ASL \int_1^b f(x)\, dx = F(b) - F(a).$$

We show that the F in the definition is uniquely determined (up to an additive constant). Indeed suppose that function $G(x)$ and $F(x)$ satisfy ASL and $F'_{ap}(x) = G'_{ap}(x)$ almost everywhere on $[a, b]$. It follows that $(F-G)'_{ap} = 0$ a.e. on $[a, b]$. Thus $F - G$ is constant on $[a, b]$. Consequently

$$\text{ASL} \int_a^b f(x)\, dx = F(b) - F(a) = G(b) - G(a).$$

This proves that the ASL-integral is well defined.

Theorem 3 *A function $f : [a, b] \to \mathbb{R}$ is SH-integrable on $[a, b]$ if and only if f is ASL-integrable on $[a, b]$.*

PROOF. Suppose first that the function f is SH-integrable on $[a, b]$. Let $F(x) = SH \int_a^x f(x)\,dx$. Then $F'_{ap}(x) = f(x)$ a.e. on $[a, b]$ and F satisfies ASL (cf, 3, Theorem 11 and Theorem 15]).

Conversely suppose that f is ASL-integrable on $[a, b]$. Then there exists an ASL function F on $[a, b]$ such that $F'_{ap}(x) = f(x)$ a.e. on $[a, b]$. We claim that f is SH-integrable on $[a, b]$ with primitive F. It is sufficient to show that F is ACGs on $[a, b]$ (cf, 3, Theorem 17]).

Let $E_n = \{x : |F'_{ap}(x)| \leq n\}$ and $H = [a, b] \setminus \bigcup_{n=1}^{\infty} E_n$, then $[a, b] = \bigcup_{n=1}^{\infty} E_n \bigcup H$. Since F is approximately differentiable on E_n, there exists a distribution S and function γ on E_n such that for every $x \in E_n, \gamma \in S_x$ and $|x - y| < \gamma(x)$, we have $|F(y) - F(x)| < n|x - y|$. This proves that F is ACs on E_n. Note that $mH = 0$, the ASL condition of F will take care of that F is ACs on H. □

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