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VANISHING DERIVATIVES AND NILPOTENCY

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, and let $t_0 \in \mathbb{R}$. We ask some basic questions concerning the derivatives $f^{(p)}$ of f at t_0. For example, does $f^{(p)}(t_0) = 0$ for all p? (Equivalently, is the Taylor series of f, expanded at t_0, constant?) If not, then what is the smallest integer p for which $f^{(p)}(t_0) \neq 0$? Our purpose here is to give precise algebraic answers to these questions.

We will answer these questions by defining an extension $\mathbb{R}^*$ of $\mathbb{R}$ and studying the behavior of a certain mapping $f^* : \mathbb{R}^* \to \mathbb{R}^*$ induced by f. Over the years, other extensions of $\mathbb{R}$ have been proposed, most notably Robinson's non-standard real line ${}^*\mathbb{R}$ [7]. Our algebra $\mathbb{R}^*$ differs from ${}^*\mathbb{R}$ in that $\mathbb{R}^*$ is not a field; it possesses zero-divisors. Although one generally expects a number system to be a field, it will become evident that it is advantageous to have zero-divisors. For example, we will answer the questions posed above in terms of the vanishing of f^* on certain sets of nilpotent elements of $\mathbb{R}^*$. Such results, of course, cannot be formulated in a field.

See [1,5,6,8] for other extensions of $\mathbb{R}$ or other results of possible related interest.

2. The Algebra $\mathbb{R}^*$

Throughout this paper, sequences are denoted by boldface letters. Individual terms of a sequence are denoted by the corresponding subscripted plain letters; for example, the general term of $\mathbf{x}$ is x_n. Sequences are assumed to be infinite unless stated otherwise. Expressions involving sequences are always expanded termwise; for example, $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$ are the sequences with general terms $x_n + y_n$ and $x_n \cdot y_n$, respectively.

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We recall that the *variation* of a sequence $\mathbf{x}$ in $\mathbb{R}$ is

$$\mathrm{Var}(\mathbf{x}) = \sum_n |x_{n+1} - x_n|. \tag{V}$$

$\mathrm{Var}(\mathbf{x})$ can be finite or infinite, depending on $\mathbf{x}$. Every bounded monotonic sequence has finite variation (since the identity (V) above telescopes). All sequences of finite variation are Cauchy sequences and therefore converge in $\mathbb{R}$.

Let $\mathcal{V}$ denote the collection of all sequences in $\mathbb{R}$ of finite variation. One sees easily that if $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, then $\mathbf{x} \pm \mathbf{y} \in \mathcal{V}$ and $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \in \mathcal{V}$. Also, if $r \in \mathbb{R}$, then $r \cdot \mathbf{x} \in \mathcal{V}$. Thus, $\mathcal{V}$ is a commutative algebra over $\mathbb{R}$. Let $\mathcal{A}$ denote the set of all absolutely summable sequences in $\mathbb{R}$. Then $\mathcal{A}$ is an ideal in $\mathcal{V}$, and we form the quotient algebra $\mathbb{R}^* = \mathcal{V}/\mathcal{A}$.

Remark 1 *In our definition of $\mathbb{R}^*$, no reason has been given for considering $\mathcal{V}$ and $\mathcal{A}$. As it turns out, $\mathcal{V}$ is closely related to the class of Lipschitz mappings (see Section 3). Also, $\mathbf{x} \in \mathcal{V}$ if and only if its "derivative"*

$$\Delta\mathbf{x} = x_1, x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots$$

belongs to $\mathcal{A}$. The technique of factoring out the subgroup of derivatives of certain groups of sequences is the cornerstone of the theory of discrete analysis [3,4], to which the reader is referred for a fuller understanding of the approach taken here. In the terminology of [3], $\mathbb{R}^*$ is the "integral envelope" of $\mathbb{R}$.

If $\mathbf{x} \in \mathcal{V}$, we denote its equivalence class in $\mathbb{R}^*$ by $[\mathbf{x}]$. If $r \in \mathbb{R}$, then the constant sequence $r, r, r, \ldots$ belongs to $\mathcal{A}$ if and only if $r = 0$. It follows that $\mathbb{R}$ is canonically embedded in $\mathbb{R}^*$. We identify $\mathbb{R}$ with its canonical image in $\mathbb{R}^*$; specifically, $\mathbb{R} \subset \mathbb{R}^*$, under the identification $r = [r, r, r, \ldots]$ for all $r \in \mathbb{R}$.

Given $\mathbf{x} \in \mathcal{V}$, we define the kth *tail* of $\mathbf{x}$ to be the sequence

$$\tau_k(\mathbf{x}) = x_{k+1}, x_{k+2}, x_{k+3}, \cdots .$$

Proposition 1 *If $\mathbf{x} \in \mathcal{V}$, then $[\mathbf{x}] = [\tau_k(\mathbf{x})]$ for every positive integer k.*

PROOF. By induction, the proof reduces to the case $k = 1$. But in this case,

$$\sum_n |\tau_1(x_n) - x_n| = \sum_n |x_{n+1} - x_n| = \mathrm{Var}(\mathbf{x}) < \infty.$$

So $\tau_1(\mathbf{x}) - \mathbf{x} \in \mathcal{A}$. □

Let $\mathcal{Z} = \{\mathbf{x} \in \mathcal{V} | \lim(\mathbf{x}) = 0\}$, and define $\mathbb{I} = \{[\mathbf{x}] \in \mathbb{R}^* | \mathbf{x} \in \mathcal{Z}\}$. Then $\mathbb{I}$ is an ideal of $\mathbb{R}^*$.

Let $x \in \mathbb{R}^*$, and suppose that $\mathbf{x}$ and $\mathbf{y}$ both represent x. Then $\mathbf{x}$ and $\mathbf{y}$ converge to the same limit in $\mathbb{R}$. It follows that x has a unique decomposition as $r_x + i_x$, where $r_x = \lim(\mathbf{x}) \in \mathbb{R}$, and $i_x = x - r_x \in \mathbb{I}$. So $\mathbb{R}^*$ splits as an additive group into $\mathbb{R} \oplus \mathbb{I}$, and we write $x = (r_x, i_x)$. Addition and scalar multiplication are then performed coordinatewise; and, since $\mathbb{I}$ is an ideal,

$$x \cdot y = (r_x \cdot r_y, r_x \cdot i_y + i_x \cdot r_y + i_x \cdot i_y) \text{ for all } x, y \in \mathbb{R}^*.$$

Recall that a ring element x is *nilpotent* if $x^p = 0$ for some positive integer p. By the rule for multiplication in $\mathbb{R}^*$ displayed above, a necessary condition for $x \in \mathbb{R}^*$ to be nilpotent is that $x \in \mathbb{I}$.

Let $\mathbf{h}$ denote the harmonic sequence, that is, $h_n = 1/n$ for all n; and let $h = [\mathbf{h}] \in \mathbb{I}$. More generally, for each real $r > 0$, the sequence $\mathbf{h}^r$ converges monotonically to zero and therefore represents $h^r \in \mathbb{I}$. If $r > 1$, then $h^r = 0$; otherwise, $h^r \neq 0$. If p is an integer such that $p > 1/r$, then $(h^r)^p = 0$; so h^r is nilpotent for each $r > 0$. For every positive integer p, define $N^p = \{x \in \mathbb{I} | x^p = 0\}$. Then $h^{1/p} \in N^{p+1} \backslash N^p$.

It follows from Theorems 1 and 3 below that non-nilpotent elements of $\mathbb{I}$ exist, but to appeal to an existence proof for this is overkill; the reader should have no trouble finding a sequence that represents a non-nilpotent element of $\mathbb{I}$.

Finally, we remark that the order relation on $\mathbb{R}$ induces a partial ordering on $\mathbb{R}^*$. Specifically, for $x, y \in \mathbb{R}^*$, we define $x \leq y$ if there exist representatives $\mathbf{x}$ and $\mathbf{y}$ of x and y, respectively, such that $x_n \leq y_n$ for all n. It can be shown that this is a well-defined partial ordering under which $\mathbb{R}^*$ is a lattice. One sees easily that if $x \in \mathbb{I}$ and if $0 < r \in \mathbb{R}$, then $x < r$. We can therefore regard $\mathbb{I}$ as the set of infinitesimals of $\mathbb{R}^*$. Our theorems will then reduce questions about the "infinitesimal calculus" to questions about this algebra of infinitesimals.

3. Functions on $\mathbb{R}^*$ Induced by Lipschitz Mappings on $\mathbb{R}$

Let X be a non-empty subset of $\mathbb{R}$. We recall that a function $f : X \to \mathbb{R}$ is a *Lipschitz mapping* if for each $x \in X$ there exist a neighborhood U of x and a real number M such that $|f(y) - f(z)| \leq M \cdot |y - z|$ for all $y, z \in U$. Every continuously differentiable function is a Lipschitz mapping, and every Lipschitz mapping is continuous. The function $|x|^{1/2}$, defined on $X = \mathbb{R}$, is the conventional example of a continuous function that is not a Lipschitz mapping.

The following proposition is immediate.

Proposition 2 *If X is closed (and therefore complete) and if $f \in$ LIP, then* $\mathrm{Var}(f(\mathbf{x}))$ *is finite whenever* $\mathrm{Var}(\mathbf{x})$ *is finite.*

The hypothesis of completeness in Proposition 2 cannot be dropped. For example, $\tan(x)$, as a function from the set of rational numbers into $\mathbb{R}$, is a Lipschitz mapping that does not preserve sequences of finite variation. We remark that the converse of Proposition 2 is also true, even without the assumption that X is complete. This can be shown by a slight modification to the proof of [2; Theorem 3.1].

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping. If $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\mathbf{x} - \mathbf{y} \in \mathcal{A}$, then it follows immediately from the definition of a Lipschitz mapping and the completeness of $\mathbb{R}$ that $f(\mathbf{x}) - f(\mathbf{y}) \in \mathcal{A}$. Consequently, f induces a function $f^* : \mathbb{R}^* \to \mathbb{R}^*$ defined by $f^*([\mathbf{x}]) = [f(\mathbf{x})]$ for all $\mathbf{x} \in \mathcal{V}$.

The following propositions, where $f, g \in$ LIP, are easily verified.

Proposition 3 *$(f+g)^* = f^* + g^*$ and $(f \cdot g)^* = f^* \cdot g^*$. Also, if $s \in \mathbb{R}$, then $(s \cdot g)^* = s \cdot g^*$.*

Proposition 4 *The identity mapping on $\mathbb{R}$ induces the identity mapping on $\mathbb{R}^*$, and $(f \circ g)^* = f^* \circ g^*$.*

Proposition 5 *If $f(0) = 0$, then $f^*(\mathbb{I}) \subset \mathbb{I}$.*

4. The Action of Induced Mappings on N^p

We are now ready to establish our results relating the vanishing of derivatives to nilpotency. Some of our proofs will involve extensions of methods employed in [2].

We are given a function $f : \mathbb{R} \to \mathbb{R}$ and a point $t_0 \in \mathbb{R}$. For each $t \in \mathbb{R}$, let $g(t) = f(t + t_0) - f(t_0)$. Then $g(0) = 0$ and $f^{(k)}(t_0) = g^{(k)}(0)$ for all k. So, replacing f by g if necessary, we can assume with no real loss of generality that $t_0 = 0$ and that $f(0) = 0$; in particular, $f^*(\mathbb{I}) \subset \mathbb{I}$.

Our first theorem follows directly from Lemmas 2 and 3 below.

Theorem 1 *If f is infinitely differentiable, then $f^{(k)}(0) = 0$ for all k if and only if $f^*(x) = 0$ for every nilpotent $x \in \mathbb{I}$.*

Definition 1 *The function f is p-smooth if f is differentiable at least p times and $f^{(p)} \in$ LIP.*

Lemma 1 *If f is p-smooth and $f^{(k)}(0) = 0$ for all $k \leq p$, then there exists a real number M and a neighborhood U of 0 such that $|f(t)| \leq M \cdot |t|^{p+1}$ for all $t \in U$.*

PROOF. Since $f^{(p)} \in$ LIP, there exist M and U such that $|f^{(p)}(t)| \leq M \cdot |t|$ for all $t \in U$. There is no loss in generality in assuming that U is an open interval.

Let $t \in U$. If $t = 0$, the equality $|f(t)| = M \cdot |t|^{p+1}$ holds trivially. Suppose then that $t \neq 0$, and put $t_0 = t$. By the Mean Value Theorem, there exists t_1 such that $0 < |t_1| < |t_0|$ and such that $f(t_0) = f'(t_1) \cdot t_0$. Inductively, for each $k \leq p$ there exists t_k such that $0 < |t_k| < |t_{k-1}|$ and such that $f^{(k-1)}(t_{k-1}) = f^{(k)}(t_k) \cdot t_{k-1}$. Then $t_p \in U$, and

$$\begin{aligned} |f(t)| &= |f'(t_1) \cdot t_0| \\ &= |f''(t_2) \cdot t_1 \cdot t_0| \\ &\quad \cdots \\ &= |f^{(p)}(t_p)| \cdot |t_{p-1} \cdot t_{p-2} \cdot \ldots \cdot t_1 \cdot t_0| \\ &\leq M \cdot |t_p \cdot t_{p-1} \cdot t_{p-2} \cdot \ldots \cdot t_1 \cdot t_0| \\ &\leq M \cdot |t|^{p+1}. \end{aligned}$$

□

Lemma 2 *If f is p-smooth and $f^{(k)}(0) = 0$ for all $k \leq p$, then $f^*(x) = 0$ for every $x \in N^{p+1}$.*

PROOF. By Lemma 1, there exist M and U such that $|f(t)| \leq M \cdot |t|^{p+1}$ for all $t \in U$. Let $x \in N^{p+1}$, and let $\mathbf{x} \in \mathcal{Z}$ represent x. By Proposition 1, we can suppose that $x_n \in U$ for all n. Then $\sum_n |f(x_n)| \leq M \cdot \sum_n |x_n|^{p+1} < \infty$. So $f(\mathbf{x}) \in \mathcal{A}$; equivalently, $f^*(x) = 0$. □

Remark 2 *The hypothesis that $f^{(p)} \in$ LIP is critical in this lemma and also in Theorem 2 below. For example, take $p = 1$ and let $f(t) = |t|^{3/2}$. Then $f^*(h^{2/3}) = h \neq 0$, even though $h^{2/3} \in N^2$.*

Lemma 3 *Let f be p-smooth, and suppose that $f^{(k)}(0) = 0$ for all $k < p$ but that $f^{(p)}(0) \neq 0$. Then $f^*(x) \neq 0$ for every $x \in N^{p+1} \backslash N^p$.*

PROOF. Let $g(t) = f(t) - C \cdot t^p$, where $C = f^{(p)}(0)/p!$. Then $g^{(k)}(0) = 0$ for all $k \leq p$. Let $x \in \mathbb{I}$. Then $f^*(x) = g^*(x) + C \cdot x^p$. If $x \in N^{p+1}$, then by Lemma 2, $g^*(x) = 0$. If $x \notin N^p$, then $C \cdot x^p \neq 0$. So $f^*(x) \neq 0$ if $x \in N^{p+1} \backslash N^p$. □

The simplest function that satisfies the hypothesis of Lemma 3 is $f(t) = t^p$. Lemma 3 ensures only that $f^*(x) \neq 0$ for every $x \in N^{p+1} \backslash N^p$, but for this particular function we obviously have $f^*(x) \neq 0$ for every $X \in \mathbb{I} \backslash N^p$. Our next theorem, which follows from Lemma 6 and Lemma 2 (applied to $p - 1$), shows that the non-vanishing of f^* throughout $\mathbb{I} \backslash N^p$ is true in general.

Theorem 2 *Let f be p-smooth, and suppose that $f^{(k)}(0) = 0$ for all $k < p$ but that $f^{(p)}(0) \neq 0$. If $x \in \mathbb{I}$, then $f^*(x) = 0$ if and only if $x \in N^p$.*

Lemma 4 *If $\mathbf{x} \in \mathcal{Z}\backslash\mathcal{A}$, then for any real numbers $0 \leq s < t$, there exists a finite subsequence $y_1, \ldots, y_m$ of $\mathbf{x}$ such that $s \leq \sum_{j \leq m} |y_j| \leq t$.*

PROOF. Remove from $\mathbf{x}$ all terms x_n such that $|x_n| > t - s$, and call the resulting subsequence $\mathbf{y}$. Since only finitely many terms of $\mathbf{x}$ have been removed, $\mathbf{y} \notin \mathcal{A}$; this implies the existence of an index m such that $s \leq \sum_{j \leq m} |y_j|$. If m is chosen to be the smallest such index, then, since $|y_m| \leq t - s$, this sum is bounded above by t. □

Lemma 5 *If $\mathbf{x} \in \mathcal{Z}$ represents $x \in \mathbb{I}\backslash N^p$, then there exists a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $[\mathbf{y}] \in N^{p+1}\backslash N^p$.*

PROOF. Since $\mathbf{x}$ converges to zero, there is an index m such that $|x_n| \leq 1$ for all $n > m$. Then $\sum_{n>m} |x_n|^p = \infty$; so, by Lemma 4, we can extract from $\tau_m(\mathbf{x})$ a finite subsequence $\mathbf{y}_1 = y_{1,1},\ y_{1,2}, \ldots, y_{1,m_1}$ such that $1 \leq \sum_{j \leq m_1} |y_{1,j}|^p \leq 2$. Additionally, $|y_{1,j}| \leq 1$ for all j. Inductively, for $n > 1$, extract from $\mathbf{x}$ a finite sequence $\mathbf{y}_n = y_{n,1},\ y_{n,2}, \ldots, y_{n,m_n}$ such that $y_{n,1}$ follows $y_{n-1,m_{n-1}}$ in $\mathbf{x}$, and such that

$$1/n \leq \sum_{j \leq m_n} |y_{n,j}|^p \leq 2/n. \tag{a}$$

and

$$|y_{n,j}| \leq 1/n \text{ for all } j. \tag{b}$$

We then have

$$\sum_{j \leq m_n} |y_{n,j}|^{p+1} \leq (1/n) \cdot \sum_{j \leq m_n} |y_{n,j}|^p \leq (1/n) \cdot (2/n) = 2/n^2. \tag{c}$$

The juxtaposed sequence $\mathbf{y} = \mathbf{y}_1\mathbf{y}_2\mathbf{y}_3 \ldots$ is then a subsequence of $\mathbf{y}$. The first inequality in (a) implies that $[\mathbf{y}] \notin N^p$, and (c) implies that $[\mathbf{y}] \in N^{p+1}$. □

Lemma 6 *Let f be p-smooth, and suppose that $f^{(k)}(0) = 0$ for all $k < p$ but that $f^{(p)}(0) \neq 0$. Then $f^*(x) \neq 0$ for every $x \in \mathbb{I}\backslash N^p$.*

PROOF. Let $\mathbf{x} \in \mathcal{Z}$ represent $x \in \mathbb{I}\backslash N^p$. By Lemma 5, there exists a subsequence $\mathbf{y}$ of $\mathbf{x}$ such that $y = [\mathbf{y}] \in N^{p+1}\backslash N^p$. By Lemma 3, $f^*(y) \neq 0$. Since $f(\mathbf{y})$ is a subsequence of $f(\mathbf{x})$, it follows that $f^*(x) \neq 0$. □

5. The Faithfulness of the Induced Mapping

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping such that $f(0) = 0$. We conclude this study by showing that f^* represents f "faithfully" on $\mathbb{I}$.

Theorem 3 *$f^*(x) = 0$ for every $x \in \mathbb{I}$ if and only if $f(t) = 0$ for every t in some neighborhood of zero.*

PROOF. It is immediate from Proposition 1 that if f vanishes on some neighborhood of zero, then f^* vanishes on $\mathbb{I}$. To establish the converse, suppose that f does not vanish on any neighborhood of zero. Then for every n there exists $x_n \in \mathbb{R}$ such that $|x_n| \leq 1/2^n$ and such that $f(x_n) \neq 0$. Choose an integer $j_n \geq 1/|f(x_n)|$, and let $\mathbf{z} = x_1, \ldots x_1, x_2, \ldots x_2, \ldots x_n, \ldots x_n, \ldots$ where x_n appears j_n times. Then $\mathbf{z} \in \mathcal{Z}$, and it follows immediately from the definition of j_n that $f(\mathbf{z}) \notin \mathcal{A}$; equivalently, $f^*([\mathbf{z}]) \neq 0$. □

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