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K. G. Johnson, Department of Mathematics, P.O. Box 48043, T. K. 13201 Petroupolis, Athens, Greece

THE σ-ALGEBRA GENERATED BY THE JORDAN SETS IN $\mathbb{R}^n$

Abstract

Let I be a non-degenerate interval in $\mathbb{R}^n$; J, B and L are the Jordan, Borel and Lebesgue sets respectively in I; $\sigma(J)$ is the σ-algebra generated by J; $F_0 = \{n : n$ is a subset of an F_σ subset of I having Lebesgue measure zero$\}$; $S_0 = \{b \cup n : b \in B, n \in F_0\}$; $S_1 = \{b \cup n : b \in B, n$ is a first category subset of I having Lebesgue measure zero $\}$. The symbol "$\subset$" denotes "proper subset".

THEOREM 1. $\sigma(J) = S_0$.

THEOREM 2. $B \subset \sigma(J) \subset S_1$.

1. INTRODUCTION AND JORDAN SET PRIMER

A very general theorem due to R. D. Mauldin [4] shows that S_0(defined in abstract above) is the measure ring for the collection of real almost everywhere continuous functions on an interval I. Here, however, our motivation was different. Not knowing of Mauldin's paper (before the referee so informed us) we asked, "What does the σ-algebra generated by the Boolean ring J of Jordan sets in an interval I look like?" The answer is in theorem 1 which we prove using an elementary set-theoretic approach. One could prove theorem 1 differently by first proving, "The σ-algebra generated by the Jordan sets is the measure ring of μ almost everywhere continuous real functions", then invoking Mauldin's result, but the proof would be less elementary. Theorem 1 raises the question, "Is $\sigma(J) = S_1$?", and the fact that the answer to this question is "no" is contained in Theorem 2.

Key Words: Bernstein set, Borel set, first category set, F_σ set, G_δ set, Jordan set, Lebesgue measure, Lebesgue set, second category set

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A question which the author has not answered is, "Where does $\sigma(J)$ stand relative to the analytic sets in I and relative to the universally measurable sets in I?"

The following notational conventions will be used: Let $a_i \leq b_i$ $(i = 1, \ldots, n)$. By an interval in $\mathbb{R}^n$ we mean the set of points $(x_1, \ldots, x_n)$ such that $(R)\, a_i \leq x_i \leq b_i$ or the set of points which is characterized by (R) with any or all of the $\leq$ signs replaced by $<$. In the latter circumstance the values $\pm\infty$ are possible for a_i, b_i so in particular $\mathbb{R}^n$ is an interval. In this paper m is the classical Borel measure in $\mathbb{R}^n$ which extends length $(n = 1)$, area $(n = 2)$, or volume $(n > 2)$ of an interval to the least σ-algebra of subsets of $\mathbb{R}^n$ containing all the intervals of $\mathbb{R}^n$. The elements of this σ-algebra are the Borel sets. A subset s of $\mathbb{R}^n$ is a Lebesgue set if and only if $s = b \cup n$ where b is a Borel set and n is a subset of a Borel set of Borel measure zero. Lebesgue measure μ of s is defined then by $\mu(s) = m(b)$ and is called the completion of m. A cube in $\mathbb{R}^n$ is an interval which has $b_i - a_i$ the same value for all i. Two cubes are said to be non-overlapping if and only if their intersection is a subset of the boundary of each.

Definition 1 *A bounded subset E of $\mathbb{R}^n$ is a Jordan set if and only if for any $\epsilon > 0$ there are sets D, F such that $D \subseteq E \subseteq F$ where D and F are each a union of finitely many pairwise non-overlapping closed cubes with edges parallel to the coordinate axes, and volume(F) - volume(D) $< \epsilon$. E is also said to be Jordan measurable.*

Volume(D) of course is the sum of the volumes of the non-overlapping cubes whose union forms D. The same is true for volume(F).

Folk Theorem 1 *Let E be a bounded subset of $\mathbb{R}^n$. Then E is Jordan measurable if and only if μ(boundary(E)) $= 0$.*

PROOF. Suppose first that E is Jordan measurable. Let $\epsilon > 0$ be given. Then $\exists$ sets D and F as in the definition such that volume(F)$-$volume(D) $< \epsilon$. Then volume(D) $= \mu$(interior(D)) $\leq \mu$(interior(E)) $\leq \mu$(closure(E)) $\leq$ volume(F). Thus μ(boundary(E)) $< \epsilon$, and since ϵ is arbitrary and positive, it follows that μ(boundary(E)) $= 0$.

Conversely, let E be bounded with μ(boundary(E)) $= 0$. Enclose E in a closed cube K with edges parallel to the coordinate axes such that closure(E) is contained in interior(K). Let $\epsilon > 0$ be given. $\exists$ open set U with boundary(E) $\subset U \subset K$ and $\mu(U) < \epsilon$. Let d be the distance from boundary(E) to $\mathbb{R}^n \setminus U$. Subdivide K into a finite number of non-overlapping congruent closed cubes having their edges parallel to the coordinate axes and such that the diameter of each of these subdividing cubes, say d', is less than d.

If K' is one of the subdividing cubes which contains an element of boundary(E) then K' is a subset of U. Let K be any subdividing cube which contains elements of interior(E) while containing no element of boundary(E). $K'' \subset$ interior(E). If not, then K'' will contain an element y outside closure(E) since K'' contains no element of boundary(E). Let x be an element of interior(E) in K''. Let $[x, y]$ be the line segment connecting x and y in K''. Then $[x, y]$ must contain an element w of boundary(E) because interior(E) and $\mathbb{R}^n \backslash$closure(E) are disjoint open sets. Because of convexity, w is in K''. A contradiction which establishes the claim. Let D be the union of all the subdividing cubes of type K'' just described and let F be the union of all of these together with all the subdividing cubes of type K' as described above. Then volume(F)$-$ volume(D) $< \epsilon$ and D and F fulfill all the requirements in the definition of Jordan measurable, and the proof is completed.

The Jordan content of E, $j(E) =$ sup volume(D) $=$ inf volume(F) for all D and F as described above for E. Then $j(E) = \mu(E)$. But $j(E)$ is an older notion than $\mu(E)$. The collection of all Jordan sets in a fixed interval I, denoted by J, is always a Boolean ring and is a Boolean algebra which contains I if and only if I is bounded. There is even a theory of measurability of functions with respect to the Jordan ring J in the following sense, "A real function f defined on I is Jordan measurable on I if at least one of the sets

$$A_\alpha = \{x : f(x) > \alpha\}, B_\alpha = \{x : f(x) \leq \alpha\}$$

is Jordan measurable for each α, with the possible exception of a set of at most countably many values of α." It turns out that a real bounded function f on a compact I is Jordan measurable if and only if f is Riemann integrable. See [1] and [3] for details.

2. PROOF OF THEOREM 1.

Let I be a fixed non-degenerate interval in $\mathbb{R}^n$. Let B and J denote the Borel and Jordan sets in I. The bounded subintervals in I are in J and they generate B. So B is a subset of $\sigma(J)$. Cardinality(B) $= c$ [2]. Since I is nondegenerate it contains a Cantor set, say s', of measure zero and every subset s of s' must be in J since μ(boundary(s)) $= 0$. Hence cardinality(B) $<$cardinality($\sigma(J)$) and $B \subset \sigma(J)$. Let $n \in F_0$. Then $n = \bigcup_{i=1}^{\infty} n_i$ such that μ(closure(n_i)) $= 0$ for all i. Thus each $n_i \in J$ and $n \in \sigma(J)$. Clearly then (*) $S_0 \subseteq \sigma(J)$. By Folk Theorem 1 any Jordan set is the union of an open set and a set whose closure is of μ measure zero. Thus (**) J $\subseteq S_0$. From (*) and (**) theorem 1 will be proved if we show S_0 to be a σ-algebra. Clearly S_0 is closed under countable unions. To show closure under complementation let $b \cup n \in S_0$, where $b \in B$,

$n \in F_0$. Then $n \subseteq n_1$ such that $\mu(n_1) = 0$ and n_1 is an F_σ set. Thus n_1 is Borel and any subset of n_1 is an element of F_0. Denote the complement of any set s by $c(s)$. Then $c(b \cup n) = c(b) \cap c(n)$ and $c(n) = c(n1) \cup (c(n) \cap n1)$ since $n \subseteq n_1$. Thus $c(b \cup n) = (c(b) \cap c(n_1)) \cup (c(b) \cap c(n) \cap n_1)$. But $c(b) \cap c(n_1) \in B$ and $c(b) \cap c(n) \cap n_1 \in F_0$. Thus $c(b \cup n)$ is in S_0 and S_0 is a σ-algebra. Thus the theorem is proved.

3. PROOF OF THEOREM 2.

The fact that $\sigma(J) \subset S_1$ is all that remains to be proved. For this consider a Bernstein set β in $\mathbb{R}^n$ [5] which has the property that both β and $c(\beta)$ have nonvoid intersection with every uncountable Borel set. Let I be a fixed non-degenerate interval and let k be a perfect nowhere dense subset of I such that every portion of k (intersection of k with an open set) has positive Lebesgue measure. So a countable subset of k is 1^{st} category in k. Let γ be a dense G_δ set in k such that $\mu(\gamma) = 0$. Then γ is 2^{nd} category in k. Let $n = \beta \cap \gamma$. Then $\mu(n) = 0$ and since $n \subseteq k$, $(*)\ n \in S_1$. But (**) n is 2^{nd} category in k. Otherwise $n \subseteq f$ where f is a 1^{st} category F_σ set in k and $\gamma \setminus f$ is 2^{nd} category in k thus uncountable and $\beta \cap (\gamma \setminus f) \neq \emptyset$ contradicting $\beta \cap \gamma \subseteq f$. To show that $n \notin S_0$ suppose the contrary. Then $n = b_1 \cup (\bigcup_{i=1}^{\infty} n_i)$ where b_1 is Borel and each $\mu(\text{closure}(n_i)) = 0$. Then b_1 is countable because otherwise $c(\beta)$ will have non-void intersection with it and $b_1 \subseteq \beta$, a contradiction. Thus b_1 is 1^{st} category in k as are each of the n_i since their closures contain no portion $(\mu(\text{closure}(n_i)) = 0)$. Thus n is 1^{st} category in k contradicting statement (**) above. So $n \notin S_0$, and S_1 is not a subset of S_0. Clearly S_0 is a subset of S_1 however, and theorem 2 now follows from theorem 1.

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