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# AN ACG FUNCTION WHICH IS NOT AN ACGs FUNCTION

## 1. Introduction and preliminaries

R. Gordon [3] asked whether an ACG function is an ACGs function. As noted by Skvortsov [8], there is an example of an ACG function that is not an ACGs function. Such a function was first constructed by Tolstov [9]. He gave an example of a function that was ACG, but not an indefinite approximately continuous Perron integral. Since the approximately continuous Perron integral and the approximately continuous Henstock integral are equivalent, this function is also not an indefinite approximately continuous Henstock integral. We will present a direct proof of this fact. The advantage of this is that the property of the indefinite AH integral used below is simpler than the property of the indefinite AP integral used by Tolstov.

Moreover, this example also shows that there are Khinchine integrable functions which are not AH integrable. Suppose that $F$ is an ACG function which is not an ACGs function. Let $ADF = f$, then $f$ is Khinchine integrable. We claim that $f$ is not AH integrable. Suppose that $f$ is AH integrable. Put $G(x) = (AH)\int_a^x f = (AP)\int_a^x f$. Since the Kubota integral [4] includes the Khinchine integral and AP integral as special cases, we have that $F$ and $G$ are the indefinite Kubota integral of $f$. Therefore, $F$ and $G$ differ by a constant. This is a contradiction.

A detailed study of ACG functions, AP functions, and Henstock integrals can be found in [7], [2], and [5] respectively. In recent years many authors (see [1] and [6]) have worked on the AP integral and the AH integral. To make the

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reading easier we list a definition and the statement of a known result which will be used in the paper.

Let $P = \{([a_i, b_i]; x_i)\}$ denote a finite collection of non-overlapping tagged intervals in $[a, b]$. We call $P$ a partial division on $E \subset [a, b]$ if $x_i \in E$ for each $i$. The following definition and lemma can be found in [3].

**Definition 1** *A distribution $S$ on $[a, b]$ is a collection of measurable sets $\{S_x : x \in [a, b]\}$ in $[a, b]$ such that $x \in S_x$ and $x$ is a point of density of $S_x$. For each $x \in [a, b]$, let $I_x = \{[c, d] : x \in [c, d] \text{ and } c, d \in S_x\}$ and let $\delta$ be a positive function defined on $[a, b]$. A partial division $P$ is $S$–subordinate to $\delta$ if $d - c < \delta(x)$ and $[c, d] \in I_x$ whenever $([c, d], x) \in P$.*

**Lemma 1** *Suppose that $F : [a, b] \to R$ is ACGs on $[a, b]$ and let $E \subset [a, b]$. If $m(E) = 0$, then for each $\epsilon > 0$ there exists a positive function $\delta$ on $E$ such that $\sum_i |F(b_i) - F(a_i)| < \epsilon$ for any partial division $P = \{([a_i, b_i]; x_i)\}$ which is $S$–subordinate to $\delta$ on $E$.*

## 2. An Example

Let $I = [0, 1]$ and let $C$ denote the Cantor ternary set. Let $\{I_i^n : 1 \le i \le 2^{n-1}\}$ be the collection of the $2^{n-1}$ intervals which are removed at the $n$'th stage. It is understood that the numbering $I_1^n$, $I_2^n$, $\cdots$, $I_{2^{n-1}}^n$ is in increasing order from left to right on the line. Let $c_i^n$ denote the center of the interval $I_i^n = [a_i^n, b_i^n]$ and let $F$ be the function defined on the interval $I$ by the following conditions:

(i) $F(x) = 0$ for $x \in C$;

(ii) $F(c_i^n) = 1/n$ for all $n$ and corresponding $i$;

(iii) The function $F$ is linear in each of the intervals $[a_i^n, c_i^n]$ and $[c_i^n, b_i^n]$.

It is clear that $F$ is continuous on $I$. Since $F$ is AC on $C$ and on each $I_i^n$, the function $F$ is ACG on $I$.

To verify that $F$ is not an ACGs function on $I$, we will show that $F$ does not satisfy the above lemma for $C \subset I$. Since $x$ is a point of density of $S_x$, for each $x \in I$ there exists $\eta(x) > 0$ such that

$$\frac{m(S_x \cap (x, y))}{y - x} > \frac{2}{3}$$

whenever $y \in (x, x + \eta(x))$. Let $\delta$ be any positive function defined on $C$ and (without loss of generality) assume that $\delta(x) < \eta(x)$ for each $x \in C$. For each positive integer $n$, let $C_n = \{x \in C : \delta(x) > 1/n\}$ and note that

$C = \bigcup_{n=1}^{\infty} C_n$. By the Baire Category Theorem, there exists an interval $(c, d)$ and a set $C_N$ such that $C_N$ is dense in the non-empty set $(c, d) \cap C$ and $d - c < 1/N$. Since $C \cap (c, d) \neq \emptyset$, there exist an interval $(u, v) \subset (c, d)$ and integers $j$ and $m$ such that $I_j^m$ is the open middle third of $(u, v)$. Now $(u, v)$ contains 2 intervals from the set $\{I_i^{m+1} : 1 \leq i \leq 2^m\}$, 4 intervals from the set $\{I_i^{m+2} : 1 \leq i \leq 2^{m+1}\}$ and in general, $2^k$ intervals from the set $\{I_i^{m+k} : 1 \leq i \leq 2^{m+k-1}\}$. Choose an integer $k$ such that $2^k/(m + k) > 2$ and denote the intervals by $\{(u_i, v_i) : 1 \leq i \leq 2^k\}$ in increasing order. Let $c_i$ denote the center of the interval $(u_i, v_i)$.

We claim that for each $i$, there exist $x_i \in C_N$ and $z_i \in [u_i, v_i]$ such that $u < x_1 \leq u_1$, $v_{i-1} < x_i \leq u_i$ for $i \geq 2$, $([x_i, z_i], x_i)$ is $S$–subordinate to $\delta$, and $F(z_i) > 1/2(m + k)$. We will establish the claim for $i = 2$; the proof for the other values of $i$ is quite similar. Since $C_N$ is dense in $(c, d) \cap C$, there exists a point $x_2 \in (v_1, u_2] \cap C_N$ such that $u_2 - x_2 < 0.1(c_2 - u_2)$. Since $c_2 - x_2 < 1/N < \eta(x_2)$,

$$\frac{m(S_{x_2} \cap (x_2, c_2))}{c_2 - x_2} > \frac{2}{3}.$$

Let $m_2 = (u_2 + c_2)/2$. If $S_{x_2} \cap (m_2, c_2) = \emptyset$, then

$$\begin{aligned} \frac{m(S_{x_2} \cap (x_2, c_2))}{c_2 - x_2} &\leq \frac{m_2 - x_2}{c_2 - x_2} = \frac{1}{2}\Big(1 + \frac{u_2 - x_2}{c_2 - x_2}\Big) \\ &\leq \frac{1}{2}\Big(1 + \frac{u_2 - x_2}{c_2 - u_2}\Big) < \frac{1}{2}\Big(1 + \frac{1}{10}\Big) < \frac{2}{3}, \end{aligned}$$

a contradiction. Let $z_2 \in S_{x_2} \cap (m_2, c_2)$. Since $F$ is increasing on $[u_2, c_2]$,

$$F(z_2) > F(m_2) = \frac{1}{2(m + k)}.$$

The partial division $\{([x_i, z_i], x_i) : 1 \leq i \leq 2^k\}$ is $S$–subordinate to $\delta$ on $C$ and

$$\sum_{i=1}^{2^k}(F(z_i) - F(x_i)) > \sum_{i=1}^{2^k} \frac{1}{2(m + k)} > 1.$$

Since $\delta$ was an arbitrary positive function defined on $C$, the function $F$ is not ACGs on $I$.

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