

Arnold W. Miller*, Department of Mathematics, University of Wisconsin, Madison, WI 53706, miller@math.wisc.edu

MEASURABLE RECTANGLE

Abstract

We give an example of a measurable set $E \subseteq \mathbb{R}$ such that the set $E' = \{(x, y) : x + y \in E\}$ is not in the σ -algebra generated by the rectangles with measurable sides. We also prove a **stronger** result that there exists an analytic (Σ_1^1) set E such that E' is not in the σ -algebra generated by rectangles whose horizontal side is measurable and vertical side is arbitrary. The same results are true when measurable is replaced with property of Baire.

The σ -algebra generated by a family \mathcal{F} of subsets of a set X is the smallest family containing \mathcal{F} and closed under taking complements and countable unions. In Rao [12] it is shown that assuming the Continuum Hypothesis every subset of the plane \mathbb{R}^2 is in the σ -algebra generated by the abstract rectangles, i.e. sets of the form $A \times B$ where A and B are arbitrary sets of reals. In Kunen [5] it is shown that it is relatively consistent with ZFC that not every subset of the plane is in the σ -algebra generated by the abstract rectangles. He shows that this is true in the Cohen real model. It also follows from a result of Rothberger [14] that if for example $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_{\omega_2}$, then not every subset of the plane is in the σ -algebra generated by the abstract rectangles. For a proof of these results see Miller [11] (remark 4 and 5 page 180).

A set is analytic or Σ_1^1 iff it is the projection of a Borel set. Answering a question of Ulam, Mansfield [7][8] showed that not every analytic subset of the plane is in the σ -algebra generated by the analytic rectangles. Note that a rectangle $A \times B \subseteq \mathbb{R} \times \mathbb{R}$ is analytic iff both A and B are analytic.

He did this by showing that, in fact, any universal analytic set is not in the σ -algebra generated by the rectangles with measurable sides. This does the trick because analytic sets are measurable (see Kuratowski [6]). This theorem was also proved by Rao [13]. Their argument shows a little more, so we give

*Research partially supported by NSF grant DMS-9024788.

Mathematical Reviews subject classification: Primary: 04A15, Secondary: 28A05

Received by the editors January 6, 1993

it next. A set $U \subseteq \mathbb{R}^2$ is a universal analytic set iff it is analytic and for every analytic set $A \subseteq \mathbb{R}$ there exist a real x such that

$$A = U_x = \{y : \langle x, y \rangle \in U\}.$$

Theorem 1 (*Mansfield [8] and Rao [13]*) *Suppose U is a universal analytic set, then U is neither in the σ -algebra generated by rectangles of the form $A \times B$ with $A \subseteq \mathbb{R}$ arbitrary and $B \subseteq \mathbb{R}$ measurable; nor in the σ -algebra generated by rectangles of the form $A \times B$ with $A \subseteq \mathbb{R}$ arbitrary and $B \subseteq \mathbb{R}$ having the property of Baire.*

PROOF. For any set U in the σ -algebra generated by rectangles of the form $A \times B$ with $A \subseteq \mathbb{R}$ arbitrary and $B \subseteq \mathbb{R}$ measurable there is a countable family $\{A_n \times B_n : n \in \omega\}$ such that each B_n is measurable and U is in the σ -algebra generated by $\{A_n \times B_n : n \in \omega\}$.

Let Z be a measure zero set and C_n be Borel sets such that for every n we have $B_n \Delta C_n \subseteq Z$ where Δ is the symmetric difference. Since Z is a measure zero set its complement must contain a perfect set P , i.e. a set homeomorphic to the Cantor space 2^ω . Now for any real x and any set V in the σ -algebra generated by $\{A_n \times B_n : n \in \omega\}$ we have that $V_x \cap P$ is Borel. This is proved by noting that it is trivial if $V = A_n \times B_n$ (since we have $P \cap B_n = P \cap C_n$), and it is preserved when taking complements and countable unions. But the set P being perfect must contain a subset A which is analytic but not Borel. Since U is universal for some real x we have that $U_x = A$ and $U_x \cap P$ is not Borel. A similar proof works for the σ -algebra generated by sets of the form $A \times B$ where B has the property of Baire, since if B has the property of Baire, then for some G , an open set, $B \Delta G$ is meager. \square In Miller [10] it is shown that it is relatively consistent with ZFC that no universal analytic set is in the σ -algebra generated by the abstract rectangles, answering a question raised by Mansfield.

James Kuelbs raised the following question¹: If $E \subseteq \mathbb{R}$ is measurable, is then

$$E' = \{(x, y) : x + y \in E\}$$

in the σ -algebra generated by the rectangles with measurable sides?

One can think of E' as a parallelogram tipped 45 degrees, so it is clear by rotation and dilation that E' is measurable if E is. Note also that since E' is the continuous preimage of E , if E is Borel then E' is Borel also.

We give a negative answer to Kuelbs' question by showing:

¹I want to thank Walter Rudin for telling me about this question and also for encouraging me to write up the solution.

Theorem 2 *For any set $E \subseteq \mathbb{R}$ we have that E is Borel iff E' is in the σ -algebra generated by rectangles of the form $A \times B$ where A and B are measurable. Similarly, E is Borel iff E' is in the σ -algebra generated by rectangles of the form $A \times B$ where A and B have the property of Baire.*

PROOF. Suppose $E' = \sigma\langle X_n \times Y_n : n \in \omega \rangle$, where X_n and Y_n are measurable for every n . Here σ is a recipe which describes how a particular set is built up (using countable intersections and complementation), i.e., it is the Borel code of E' . Since X_n and Y_n are measurable we can obtain A_n and B_n Borel sets and Z a Borel set of measure zero such that:

$$X_n \Delta A_n \subseteq Z \text{ and } Y_n \Delta B_n \subseteq Z \text{ for each } n \in \omega.$$

Claim. $u \in E$ iff $\exists x, y \notin Z$ ($x + y = u$ and $\langle x, y \rangle \in \sigma\langle A_n \times B_n : n \in \omega \rangle$).

The implication \leftarrow is clear because if $x, y \notin Z$, then since $[x \in X_n \text{ iff } x \in A_n]$ and $[y \in Y_n \text{ iff } y \in B_n]$ we have that $\langle x, y \rangle \in \sigma\langle A_n \times B_n : n \in \omega \rangle$ iff $\langle x, y \rangle \in \sigma\langle X_n \times Y_n : n \in \omega \rangle$ and hence $u \in E$.

The implication \rightarrow is true because of the following argument. Suppose $u \in E$ is given. Choose $x \notin Z \cup (u - Z)$. Since these are measure zero sets this is easy to do. But now let $y = u - x$, then $y \notin Z$ since $x \notin u - Z$. Since $u \in E$ it must be that $\langle x, y \rangle \in E'$ and so $\langle x, y \rangle \in \sigma\langle X_n \times Y_n : n \in \omega \rangle$ and thus $\langle x, y \rangle \in \sigma\langle A_n \times B_n : n \in \omega \rangle$. This proves the Claim.

By the Claim, E is the projection of a Borel set and hence analytic. But note that $(E^c)' = (E')^c$ where E^c denotes the complement of E . It follows that E^c is also analytic and so by the classical theorem of Souslin, E is Borel (see Kuratowski [6]).

A similar proof works for the Baire property since we need only that every set with the property of Baire differs from some Borel set by a meager set. \square

This answers Kuelbs' question since if E is analytic and not Borel (or any measurable set which is not Borel), then E' is not in the σ -algebra generated by rectangles with measurable sides. The argument also shows, for example, that a set $E \subseteq \mathbb{R}$ is analytic iff E' can be obtained by applying operation A to the σ -algebra generated by the rectangles with measurable sides.

Next we show that a slightly stronger result holds for the sets of the form E' where E is analytic. The following lemma is the key.

Lemma 3 *There exists an analytic set $E \subseteq \mathbb{R}$ such that for any Z which has measure zero or is meager there exists $x \in \mathbb{R}$ such that $E' \setminus (x + Z)$ is not Borel.*

PROOF. Note that we may construct two such sets, one for category and one for measure, and then putting them into disjoint intervals and taking the union would suffice to prove the lemma.

We first give the proof for category. We may assume that the set Z is the countable union of compact nowhere dense sets. This is because, if Z' is any meager set, then there is such a $Z \supseteq Z'$. But now if $E \setminus (x + Z)$ is not Borel, then neither is $E \setminus (x + Z')$ since $E \setminus (x + Z) = (E \setminus (x + Z')) \setminus (x + Z)$.

It is a classical result that the set of irrationals is homeomorphic to the Baire space, ω^ω , which is the space of infinite sequences of integers with the product topology (see Kuratowski [6]). Let $h : \mathbb{R} \setminus \mathbb{Q} \rightarrow \omega^\omega$ be a homeomorphism. For $f, g \in \omega^\omega$ define $f \leq^* g$ iff for all but finitely many $n \in \omega$ we have $f(n) \leq g(n)$. It is not hard to see that for any countable union of compact sets $F \subseteq \omega^\omega$ there exists $f \in \omega^\omega$ such that $F \subseteq \{g \in \omega^\omega : g \leq^* f\}$. Therefore, if $G \subseteq \mathbb{R}$ is a G_δ set (countable intersection of open sets) which contains the rationals, then $\mathbb{R} \setminus G$ is a σ -compact subset of $\mathbb{R} \setminus \mathbb{Q}$ and therefore there exists $f \in \omega^\omega$ such that for every $g \geq^* f$ we have $h^{-1}(g) \in G$. (This is a trick going back at least to Rothberger [14]).

For $p : \omega^\omega \rightarrow 2^\omega$ (the parity function) by

$$p(g)(n) = \begin{cases} 0 & \text{if } g(n) \text{ is even} \\ 1 & \text{if } g(n) \text{ is odd} \end{cases}$$

Let $E_0 \subseteq 2^\omega$ be an analytic set which is not Borel. Define

$$E = \{g \in \mathbb{R} \setminus \mathbb{Q} : p(h(g)) \in E_0\}.$$

Now suppose that $Z \subseteq \mathbb{R}$ is a meager set which is the countable union of compact sets and let $G = \mathbb{R} \setminus Z$. Let $x_0 \in \bigcap_{q \in \mathbb{Q}} (q - G)$ be arbitrary (this set is nonempty since each $q - G$ is comeager), and note that $\mathbb{Q} \subseteq (x_0 + G)$.

Hence there exists $f \in \omega^\omega$ such that for every $g \in \omega^\omega$ with $g \geq^* f$ we have $h^{-1}(g) \in x_0 + G$. Without loss we may assume that for every $n \in \omega$ that $f(n)$ is even. Let

$$P = \{g \in \omega^\omega : \forall n \, g(n) = f(n) \text{ or } g(n) = f(n) + 1\}.$$

Then P is homeomorphic to 2^ω and

$$Q = h^{-1}(P) \subseteq x_0 + G$$

and so

$$Q \cap (x_0 + Z) = \emptyset.$$

But clearly $Q \cap E$ is homeomorphic to E_0 via $p \circ h$ and therefore $E \setminus (x_0 + Z)$ cannot be Borel.

Next we give the proof for measure. Here we use a coding technique due to Bartoszynski and Judah [1] who used it to show that a dominating real followed by a random real gives a perfect set of random reals (Theorem 2.7 [1]). Only a reader thoroughly familiar with [1] and a fan of forcing should attempt to read this. We begin by giving the proof in a slightly different situation, namely 2^ω instead of the reals and where $+$ denotes pointwise addition modulo 2 on 2^ω and the usual product measure on 2^ω . Afterwards we will indicate how to modify the proof for the reals with ordinary addition and Lebesgue measure.

Define $I \subseteq 2^\omega$ to be the set of all $x \in 2^\omega$ which have infinitely many ones and infinitely many zeros. It is easy to see that I is a G_δ set. Any $x \in I$ can be regarded as a sequence of blocks of consecutive ones, i.e., blocks of consecutive ones each separated by blocks of one or more zeros. Define $q : I \rightarrow \omega^\omega$ by $q(x)(n)$ is the length of the n^{th} block of consecutive ones. As above let $E_0 \subseteq 2^\omega$ be an analytic set which is not Borel and let $p(f)$ for $f \in \omega^\omega$ be the parity function. Let

$$E = \{x \in I : p(q(x)) \in E_0\}.$$

We claim that this works, i.e. given a measure zero set $Z \subseteq 2^\omega$ there exists a real x such that $E \setminus (x + Z)$ is not Borel. Let d_n for $n \in \omega$ be the dominating sequence as given in the proof of Theorem 2.7 [1]. Without loss we may assume that $d_{2n+2} - d_{2n+1}$ is even if n is even and odd if n is odd.

According to Bartoszynski and Judah there exist sufficiently random reals $r, r' \in 2^\omega$ such that following holds. Let r'' be defined by $r''(n) = r'(n) + 1 \bmod 2$ for each n . Define P to be the set of all $x \in 2^\omega$ such that for every n we have that

$$x \upharpoonright [d_{2n}, d_{2n+1}) = r \upharpoonright [d_{2n}, d_{2n+1})$$

and either

$$x \upharpoonright [d_{2n+1}, d_{2n+2}) = r' \upharpoonright [d_{2n+1}, d_{2n+2})$$

or

$$x \upharpoonright [d_{2n+1}, d_{2n+2}) = r'' \upharpoonright [d_{2n+1}, d_{2n+2}).$$

The main difficulty of Bartoszynski and Judah's proof is to show that the perfect set P is disjoint from Z . Let $x_1 \in 2^\omega$ be defined by

$$x_1 \upharpoonright [d_{2n}, d_{2n+1}) = r \upharpoonright [d_{2n}, d_{2n+1})$$

and

$$x_1 \upharpoonright [d_{2n+1}, d_{2n+2}) = r' \upharpoonright [d_{2n+1}, d_{2n+2}).$$

Let $Q \subseteq 2^\omega$ be the perfect set of all $x \in 2^\omega$ such that $x \upharpoonright [d_{2n}, d_{2n+1})$ is constantly zero and $x \upharpoonright [d_{2n+1}, d_{2n+2})$ is constantly zero or constantly one. It then follows that $P = x_1 + Q$ and that Q is disjoint from $x_1 + Z$. Note

that the mapping $(p \circ q)$ takes Q onto 2^ω . Hence $E \cap Q$ is not Borel, because $d_{2n+2} - d_{2n+1}$ is even if n is even and odd if n is odd and so it easy to see the E_0 is coded into $E \cap Q$.

Now we indicate how to modify the above proof so as to work for the reals with ordinary addition and Lebesgue measure. First we modify it to work for the unit interval $[0, 1]$ with ordinary addition modulo one. Let $s : 2^\omega \rightarrow [0, 1]$ be the map defined by

$$s(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

This map is continuous, measure preserving, and one-to-one except on countably many points where it is two-to-one. On the points x where it is two-to-one let us agree that $s^{-1}(x)$ denotes the preimage of x which is eventually zero. The main difficulty is that addition mod 1 in $[0, 1]$ is quite different than pointwise addition modulo 2 in 2^ω . Define for $x, y \in 2^\omega$ the operation $x \oplus y$ to be $s^{-1}(s(x) + s(y))$ where $s(x) + s(y)$ is the ordinary addition in $[0, 1]$ modulo 1. The operation \oplus just corresponds to a kind of pointwise addition with carry. Instead of r'' being the complement of r' as in the proof of Bartoszyński and Judah we will take a sparser translate. Let $Q \subseteq 2^\omega$ be the set of $x \in 2^\omega$ such that $x(m) = 1$ only if for some n we have $m = d_{2n+2} - 1$, i.e., the last element of the interval $[d_{2n+1}, d_{2n+2})$.

Note that the set of all $r \in 2^\omega$ such that for all but finitely many n there exists $i \in [d_n, d_{n+1})$ such that $r(i) = 0$ has measure one. Hence, by changing our d_n if necessary we may assume that our random real r has the property that for every n there is an $i \in [d_{2n+1}, d_{2n+2})$ such that $r(i) = 0$. This means that when we calculate $r \oplus x$ for any $x \in Q$ the carry digit on each interval $[d_{2n+1}, d_{2n+2})$ does not propagate out of that interval. Now by the argument of Bartoszyński and Judah there exists $r \in 2^\omega$ sufficiently random so that $r \oplus Q$ is disjoint from $s^{-1}(Z)$.

We also use a different coding scheme. We may assume that for every n that $d_{2n+2} - 1$ is even for even n and odd for odd n . Let J be the set of all $x \in 2^\omega$ such that there are infinitely many n with $x(n) = 1$. Let $q : 2^\omega \rightarrow 2^\omega$ be defined by $q(x) = y$ where $\{i_n : n \in \omega\}$ lists in order all i such that $x(i) = 1$ and $y(n) = 1$ iff i_n is even. Now let

$$E = \{x \in [0, 1] : q(s^{-1}(x)) \in E_0\}.$$

Let $x = s(r)$ and note that $(x + E) \setminus Z$ is not Borel, since $x + s(Q)$ is disjoint from Z . Also note that we can assume $r(0) = 0$ and so $x + s(Q)$ is the same whether we do addition or addition modulo one. But $E \setminus (-x + Z)$ is just the translate of $(x + E) \setminus Z$ via $-x$ and so we are done.

□

Theorem 4 *There exists $E \subseteq \mathbb{R}$ which is analytic (hence measurable and having the property of Baire) such that $E' = \{(x, y) : x + y \in E\}$ is not in either the σ -algebra generated by rectangles of the form $A \times B$ with A arbitrary and B measurable, nor is it in the σ -algebra generated by rectangles of the form $A \times B$ with A arbitrary and B having the property of Baire.*

PROOF.

Let E be the analytic set given by Lemma 3. Suppose for contradiction that $E' = \sigma\langle A_n \times B_n : n \in \omega \rangle$ where the A_n are arbitrary and the B_n are measurable. Let Z be a measure zero Borel set and \tilde{B}_n be Borel such that $B_n \Delta \tilde{B}_n \subseteq Z$ for every $n \in \omega$. Suppose that $E \setminus (x + Z)$ is not Borel. By translating this set by $-x$ we must have that $(-x + E) \setminus Z$ is not Borel. Define \tilde{B}_n as follows. If $-x \in A_n$ let $\tilde{B}_n = B_n$ and if $-x \notin A_n$ let $\tilde{B}_n = \emptyset$. Define $C = \sigma\langle \tilde{B}_n : n \in \omega \rangle$, i.e., C has exactly the same Borel code as $E' = \sigma\langle A_n \times B_n : n \in \omega \rangle$ except at the base we substitute \tilde{B}_n for $A_n \times B_n$. Since each \tilde{B}_n is Borel, the set C is a Borel set. Now for any $y \notin Z$ we have that

1. $y \in (-x + E)$ iff
2. $x + y \in E$ iff
3. $(x, y) \in E'$ iff
4. $(x, y) \in \sigma\langle A_n \times B_n : n \in \omega \rangle$ iff
5. $(x, y) \in \sigma\langle A_n \times \tilde{B}_n : n \in \omega \rangle$ iff
6. $y \in \sigma\langle \tilde{B}_n : n \in \omega \rangle = C$.

(4) and (5) are equivalent because $y \notin Z$. (5) and (6) are proved equivalent by an easy induction on the Borel code σ .

Consequently, for every $y \notin Z$ we have that $y \in (-x + E)$ iff $y \in C$. But this means that $(-x + E) \setminus Z = C \setminus Z$ which contradicts the assumption that $(-x + E) \setminus Z$ is not Borel (both C and Z are Borel). A similar proof works for the property of Baire case. \square

Let P be the proposition that E is Borel iff E' is in the σ -algebra generated by rectangles of the form $A \times B$ with A arbitrary and B measurable. Kunen has shown that P is false if the continuum hypothesis is true, but P is true in the random real model.

There is other work on measurable rectangles which does not seem to be directly related to this. For example, Eggleston [3] proves that every subset of the plane of positive measure contains a rectangle $X \times Y$ with X uncountable (in fact perfect) and Y of positive measure. Martin [9] gives a metamathematical proof of this result.

Erdoes and Stone [4] show that there exist Borel sets A and B such that the set $A + B = \{x + y : x \in A, y \in B\}$ is not Borel.

Friedman and Shelah (see Burke [2] or Steprans [15]) proved that in the Cohen real model for any F_σ subset E of the plane, if E contains a rectangle of positive outer measure, then E contains a rectangle of positive measure. One corollary of this is that it is consistent that there is a subset of the plane of full measure which does not contain any rectangle $A \times B$ with both A and B having positive outer measure. To see this let $E \subseteq \mathbb{R}$ be any meager F_σ set with full measure. Then $E' = \{(x, y) : x + y \in E\}$ is a subset of the plane of full measure which is meager. It cannot contain a rectangle of positive measure $A \times B$ since by the classical theorem of Steinhaus the set $A + B$ would contain an interval and hence E would not be meager.

References

- [1] T. Bartoszyński and H. Judah, *Jumping with random reals*, Annals of Pure and Applied Logic, **48** (1990), 197–213.
- [2] M. Burke, *A theorem of Friedman on rectangle inclusion and its consequences*, unpublished.
- [3] H. Eggleston, *Two measure properties of Cartesian product sets*, Quarterly Journal of Mathematics Oxford, **5** (1954), 108–115.
- [4] P. Erdős and A. H. Stone, *On the sum of two Borel sets*, Proceedings of the American Mathematical Society, **25** (1969), 304–306.
- [5] K. Kunen, *Inaccessible properties of cardinals*, Doctoral Dissertation, Stanford University, 1968.
- [6] K. Kuratowski, *Topology*, Vol I, Academic Press, New York 1966.
- [7] R. Mansfield, *The solution of one of Ulam's problems concerning analytic rectangles*, in *Axiomatic Set Theory*, Proceedings of the Symposium in Pure Mathematics, Vol. 8 part I, American Mathematical Society, (1971), 241–245.
- [8] R. Mansfield, *The solution of one of Ulam's problems concerning analytic rectangles II*, Proceedings of the American Mathematical Society, **26** (1970), 539–540.
- [9] D. A. Martin, *The use of set-theoretic hypotheses in the study of measure and topology*, General Topology and Modern Analysis edited by Rao, McAuley, Academic Press (1981), 417–430.

- [10] A. W. Miller, *On the length of Borel hierarchies*, *Annals of Mathematical Logic*, **16** (1979), 233–267.
- [11] A. W. Miller, *Generic Souslin sets*, *Pacific Journal of Mathematics*, **97** (1981), 171–181.
- [12] B. V. Rao, *On discrete Borel spaces and projective sets*, *Bulletin of the American Mathematical Society*, **75** (1969), 614–617.
- [13] B. V. Rao, *Remarks on analytic sets*, *Fundamenta Mathematicae*, **66** (1970), 237–239.
- [14] F. Rothberger, *A remark on the existence of a denumerable base for a family of functions*, *Canadian Journal of Mathematics*, **4** (1952), 117–119.
- [15] J. Steprans, *Combinatorial consequences of adding Cohen reals*, to appear.