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HOW TO OBTAIN ALL FINE CATEGORY DENSITY TOPOLOGIES

Introduction

In the eighties, Professor Wilczyński and his colleagues published a series of papers on Wilczyński's $\mathcal{I}$-density topology (cf. [7]) which is, in a sense, analogous to the density topology on the line. One of the main analogies consists of the fact that both topologies arise from some "lower density operator" and therefore have similar properties. L. Zajíček showed in [2] that the $\mathcal{I}$-density topology is created by the $*$-modification of the so-called porous topology. In addition he showed that if in a Baire topological space the $*$-modification is applied to a topology that is S-related to the original topology, then a category density topology (i.e. topology determined by a category lower density operator) is obtained as a consequence.

The main result of the present paper contained in the first part is the assertion that every fine category density topology in a Baire regular topological space can be obtained in this way; i.e. every fine category density topology is obtained as the $*$-modification of some (called "primitive" in the paper) topology, S-related to the original topology. Moreover, two ways of obtaining the coarsest and the finest of all topologies primitive to the original category density topology are presented.

In the second part of the paper, the results are applied to some well-known fine topologies in $\mathbb{R}$. A consequence of these applications is the observation that a primitive topology generally is not determined uniquely.

Preliminaries

First, let us recall several definitions, notation and facts.

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Let X be topological space, Σ the family of all subsets of X having the Baire property and $\mathcal{N}$ the family of all first-category subsets of X. We will write $A \sim B$, where $A, B \subset X$, to denote the fact that $(A \setminus B) \cup (B \setminus A) \in \mathcal{N}$.

A mapping $S : \Sigma \to \Sigma$ satisfying conditions

(S1) $S(A) \sim A$

(S2) $A \sim B \implies S(A) = S(B)$

(S3) $S(\emptyset) = \emptyset, \;\; S(X) = X$

(S4) $S(A \cap B) = S(A) \cap S(B)$

is called category lower density. If S is a category lower density and τ_S is the family of sets defined by $\tau_S = \{A \in \Sigma : A \subset S(A)\}$, then τ_S forms a topology (cf. [1]) that is called the category density topology (determined by the category lower density S). Moreover, $\tau_S = \{S(A) \setminus N : A \in \Sigma, \;\; N \in \mathcal{N}\}$.

The following theorem yields useful characterizations of category density topologies.

Theorem A (cf. [1]). *Let τ be a topology for X. Then the following are equivalent.*

(i) *τ is a category density topology.*

(ii) *τ has the following properties:*

  (a) $A \in \mathcal{N} \implies A$ *is τ-closed*

  (b) *every nonempty τ-open set contains a subset from $\Sigma \setminus \mathcal{N}$*

  (c) $A \sim int_\tau A \sim \overline{A}^\tau$ *for every* $A \in \Sigma$.

(iii) *τ has the following properties:*

  (a) $A \in \mathcal{N} \iff A$ *is τ-closed and τ-nowhere dense*

  (b) $A \in \Sigma \iff A$ *has the τ-Baire property.*

Now we will give the definitions of some particular topologies. Let (X, t) be a topological space.

A set $H \subset X$ is said to be regular open if $H = int_t \overline{H}^t$. The family of all regular open sets forms a basis of a topology (cf. [5]) that will be denoted by t^+.

The family $\{G \setminus N : G \in t, \; N$ is of the t-first category$\}$ is a topology (cf. [2]). It will be denoted by t^*.

The family of all subsets A satisfying $A \subset int_t \overline{int_t A}^t$ forms a topology (denoted by t^α) such that $t^\alpha = \{B \setminus N : B \in t, \; N$ is t-nowhere dense$\}$ (cf. [4]).

Let τ be a topology finer than t. The family of τ-open sets of the form $U \cup \{x\}$, where $U \in t$ and $x \in X$ forms a basis of a topology; we will call it the a.e.-modification of τ (cf. [1] and [6]) and denote it by τ^a.

Let t_1 and t_2 be topologies on a set X. Then t_1, t_2 are said to be S-related if for any set $A \subset X$, $int_{t_1} A \neq \emptyset$ iff $int_{t_2} A \neq \emptyset$ (cf. [2]).

Let ω be a fine category density topology on (X, t). A topology τ will be called primitive to ω if τ and t are S-related and $\tau^* = \omega$.

In what follows, it will be helpful for us to be able to describe a category lower density in terms of the category density topology determined by it.

Proposition 1. *Let τ be a category density topology determined by a category lower density S. Then for each $A \in \Sigma$, $S(A) = int_\tau \overline{A}^\tau$ and $S(A)$ is τ-regular open.*

PROOF. Let $A \in \Sigma$. Obviously, $S(A) \subset \overline{A}^\tau$ (otherwise $\emptyset \neq S(A) \setminus \overline{A}^\tau \in \tau$, which is contrary to the statement (ii)–(b) of Theorem A, since $S(A) \sim \overline{A}^\tau$). Applying Theorem A and properties of category lower density we obtain $int_\tau \overline{A}^\tau \subset S(int_\tau \overline{A}^\tau) = S(A) \subset int_\tau \overline{A}^\tau$ and therefore $S(A) = int_\tau \overline{A}^\tau$.

Consequently, since $S(S(A)) = S(A)$, we have $S(A)$ is τ-regular.

Part 1

Let us start with the theorem which has motivated and inspired this work.

Theorem B (cf. [2]). *Let (P, ϱ) be a Baire topological space and let τ be a topology on P which is S-related to ϱ. Then the topology τ^* is a category density topology on (P, ϱ) and $\tau^* = \{G \setminus N : G$ is τ-open, N is of the ϱ-first category$\}$.*

The next theorem shows that in the case of a regular topological space (i.e. a space such that between every point and a neighborhood of it some closed neighborhood can be inserted; but not necessarily a Hausdorff space) and a fine category density topology, the converse of Theorem B is also true. Otherwise stated, in a regular Baire space, to every fine category density topology there exists a primitive topology.

Theorem 2. *Let (P, ϱ) be a Baire regular topological space and let ω be a category density topology on (P, ϱ) which is finer than ϱ. Then the topology ω^+ is the coarsest of all topologies primitive to ω. Moreover, $\varrho \subset \omega^+$.*

Corollary. *If in addition each singleton in (P, ϱ) is of the ϱ-first category, then the a.e.-modification of ω is the finest of all topologies primitive to ω.*

PROOF OF THEOREM 2. Denote by S the category lower density corresponding to the topology ω, by Σ the family of all subsets of P with the ϱ-Baire property and by $\mathcal{N}$ the σ-ideal of all ϱ-first category sets. Note that it follows from Proposition 1. that the set $\{S(G) : G \in \varrho\}$ is a basis for the topology ω^+.

Now prove that ω^+ and ϱ are S-related. Let $\emptyset \neq T \in \omega^+$. Then T can be written in the form $T = \bigcup_{\alpha \in \mathcal{A}} S(G_\alpha)$, where $G_\alpha \in \varrho$ for any $\alpha \in \mathcal{A}$. Since $T \neq \emptyset$, there exists $\alpha_0 \in \mathcal{A}$ such that $G_{\alpha_0} \neq \emptyset$. From the assumption $\varrho \subset \omega$ it follows that $G_{\alpha_0} \subset S(G_{\alpha_0}) \subset T$.

For the converse, we shall prove that $\varrho \subset \omega^+$. Let $\emptyset \neq G \in \varrho$. It follows from the regularity of the space (P, ϱ) that for any $x \in G$ there exists $U_x \in \varrho$ such that $x \in U_x \subset \overline{U_x}^{\varrho} \subset G$. For every such U_x, $x \in U_x \subset S(U_x) \subset \overline{U_x}^{\omega} \subset \overline{U_x}^{\varrho} \subset G$ by Proposition 1. and the assumption of $\varrho \subset \omega$. Consequently, $G = \bigcup_{x \in G} S(U_x)$, and so $G \in \omega^+$.

Now we give the proof that $(\omega^+)^* = \omega$. According to Theorem B, we have $(\omega^+)^* = \{T \setminus N : T \in \omega^+, N \in \mathcal{N}\}$. It suffices to realize that $\omega = \{S(G) \setminus N : G \in \varrho, N \in \mathcal{N}\}$, $\omega^+ \subset \omega$ and that the members of $\mathcal{N}$ are ω-closed sets.

We have proved that the topology ω^+ is primitive to ω. It remains to prove that ω^+ is the coarsest of all topologies primitive to ω. Let τ be a primitive to ω. It suffices to prove that $S(G) \in \tau$ for every $G \in \varrho$. Let $G \in \varrho$ be given. Clearly, $\tau \subset \tau^* = \omega$. Since $S(G) \in \omega$, there exist sets $T \in \tau$ and N of the ϱ-first category such that $S(G) = T \setminus N$. In view of this and properties of category lower density, we obtain $T \subset S(T) = S(T \setminus N) = S(S(G)) = S(G) = T \setminus N \subset T$. Consequently, $T = S(G)$ and $S(G) \in \tau$.

PROOF OF COROLLARY. We begin with the proof that ω^a and ϱ are S-related. One direction of this statement is clear, for $\varrho \subset \omega^a$. To prove the converse, assume $A \subset P$ and $int_{\omega^a} A \neq \emptyset$. Thus, there exist $G \in \varrho$ and $x \in P$ such that $G \cup \{x\} \in \omega$ and $G \cup \{x\} \subset A$. Obviously, $G \neq \emptyset$ (otherwise $\{x\} \in \omega$, so $\{x\} \in \Sigma \setminus \mathcal{N}$, which is a contradiction), and so $\emptyset \neq G = int_\varrho G \subset int_\varrho A$.

To finish the proof that ω^a is primitive to ω it suffices to verify $(\omega^a)^* = \omega$. Since $\omega^a \subset \omega$ and sets of the ϱ-first category are ω-closed, $(\omega^a)^* = \{T \setminus N : T \in \omega^a, \ N \in \mathcal{N}\} \subset \omega$. The oppisite inclusion is an easy consequence of the fact that for any $G \in \varrho$ we have $S(G) = \bigcup_{x \in S(G)} (G \cup \{x\}) \in \omega^a$, which follows from $\varrho \subset \omega$ and the definition of the topology ω^a.

Now let τ be a primitive topology to ω and $T \in \tau$. To finish the proof it is sufficient to show that $T \in \omega^a$. We already know that $\varrho \subset \omega^+ \subset \tau$. Further $T \in \omega$, since $\tau^* = \omega$. Now $T \subset \overline{int_\varrho T}^{\varrho}$ since otherwise $\emptyset \neq T \setminus \overline{int_\varrho T}^{\varrho}$ and because of the assumption that the topologies τ and ϱ are S-related, there exists a nonvoid ϱ-open set U satisfying $U \subset T \setminus \overline{int_\varrho T}^{\varrho} \subset T \setminus int_\varrho T$, which is impossible. Now using the properties of category lower density we obtain

$$S(int_\varrho T) \subset S(T) \subset S(\overline{int_\varrho T}^{\varrho}) = S(int_\varrho T). \tag{1}$$

Since $T = \bigcup_{x \in T}(int_\varrho T \cup \{x\})$ and since $int_\varrho T \cup \{x\} \subset S(T)$ for each $x \in T$ $(T \in \omega)$, it follows that $T \in \omega^a$, according to (1). □

Proposition 3. describes another way of obtaining the coarsest (the finest, respectively) primitive topology under the assumption that some primitive topology is already at our disposal.

Proposition 3. *Let (P, ϱ) be a regular Baire topological space and let t be a topology finer than ϱ such that t and ϱ are S-related. Then* $(t^*)^+ = t^+$.

Corollary 1. *If in addition we assume that every singleton in (P, ϱ) is of the ϱ-first category, then moreover* $(t^*)^a = ((t^\alpha)^*)^a = t^\alpha$.

Corollary 2. *If, in addition, t is regular, then* $(t^*)^+ = t^+ = t$.

PROOF OF PROPOSITION 3. We will carry out the proof by establishing the equality of the bases of the topologies $(t^*)^+$ and t^+, i.e. by showing $\{S(G) : G \in \varrho\} = \{G \in t : G = int_t \overline{G}^t\}$ (by S a category lower density is again denoted, this time the one determined by the topology t^*). First of all, note that $S(G) = int_t \overline{G}^t$ for every $G \in t$. (Let $G \in t$. Owing to the S-relation between the topologies ϱ and t, $\overline{G}^t \setminus G$ is a ϱ-first category set and so $S(G) = S(\overline{G}^t)$. Applying Proposition 1. we get $S(G) = S(\overline{G}^t) = int_{t^*} \overline{G}^t$ from which it follows that $int_t \overline{G}^t \subset S(G)$. Conversely, it follows from Theorem 2. that $S(G) \in t$. Moreover, since $S(G) \subset \overline{G}^t$, we have $S(G) \subset int_t \overline{G}^t$.)

Lastly, we give the promised proof of equality of the bases. Let $G \in \varrho$. Then $G \in t$. Consequently $S(G) = int_t \overline{G}^t$. But also $S(G) \in t$, so $int_t \overline{S(G)}^t = S(S(G)) = S(G)$. For the converse, let $G \in t$, $G = int_t \overline{G}^t$. Then $S(G) = G$ which completes the proof, since from the properties of category density topologies it follows that $\{S(G) : G \in \varrho\} = \{S(A) : A$ has the ϱ-Baire property$\}$.

PROOF OF COROLLARY 1. First, prove the equality $(t^\alpha)^* = t^*$.

The inclusion $t^* \subset (t^\alpha)^*$ is clear, for $t \subset t^\alpha$. Conversely let $T \in t^\alpha$, i.e. there exist sets $B \in t$ and N t-nowhere dense such that $T = B \setminus N$. Thus, $T \in t^*$ and it is sufficient to realize that sets of the ϱ-first category are t^*-closed. Since the topologies t and t^α are obviously S-related, it is sufficient to prove $(t^*)^\alpha \subset t^\alpha$ (in view the Corollary to Theorem 2.). Note that for any $G \in \varrho$ $(\subset t)$ and $x \in S(G)$, $G \subset G \cup \{x\} \subset S(G) \subset \overline{G}^t$. Since $\overline{G}^t \setminus G$ is a t-nowhere dense set, $S(G) \setminus (G \cup \{x\})$ is t-nowhere dense also. Using the definition of t^α and Theorem 2. we obtain $G \cup \{x\} = S(G) \setminus (S(G) \setminus (G \cup \{x\})) \in t^\alpha$. Noting that the family $\{G \cup \{x\} : G \in \varrho,\ x \in S(G)\}$ constitutes a basis for the topology $(t^*)^a$ completes the proof

PROOF OF COROLLARY 2. We already know that $t^+ = (t^*)^+ \subset t$. Thus, it remains to prove $t \subset (t^*)^+$. Let $U \in t$. It follows from the regularity of the topology t that for every $x \in U$ there exists a set $G_x \in t$ such that $x \in G_x \subset \overline{G_x}^t \subset U$. If we use Proposition 1. and the fact that $t \subset t^*$, then for each of these sets we obtain G_x $S(G_x) = int_{t^*} \overline{G_x}^{t^*} \subset \overline{G_x}^{t^*} \subset \overline{G_x}^t \subset U$. Hence, $U = \bigcup_{x \in U} S(G_x) \in (t^*)^+$, which was to be proved.

Part 2

This part is devoted to applications of the preceding results to some well-known fine topologies in $(\mathbb{R}, e)$ ("e" means the Euclidean topology). Recall their definitions and notations.

The density topology (denoted here by d) consists of all measurable (with respect to the Lebesgue measure λ) sets M such that

$$\forall x \in M: \lim_{h \to 0+} \frac{1}{2h} \lambda(M \cap (x-h, x+h)) = 1$$

(cf. [1]).

The r-modification (cf. [1]) of the topology d (denote it by d^r) is a topology, whose basis is the family of all subsets of $\mathbb{R}$ that are d-open as well as F_σ and G_δ (with respect to e).

The definition of the p-topology (so-called porous topology, cf. [2]) is somewhat more intricate. Given $x \in \mathbb{R}$, $R > 0$, $M \subset \mathbb{R}$; by $\varphi(x, R, M)$ we mean the supremum of the set of all $r > 0$ for which there exists $y \in \mathbb{R}$ such that $U(y, r) \subset U(x, R) \setminus M$. If $\limsup_{R \to 0+} \varphi(x, R, M) \cdot R^{-1} > 0$, then M is said to be porous at x. Furthermore, a set $E \subset \mathbb{R}$ is said to be superporous at x if $E \cup F$ is porous at x for any set F that is porous at x. The p-topology is formed of all sets $G \subset \mathbb{R}$ such that $\mathbb{R} \setminus G$ is superporous at each point of G.

Recall that the topologies d^a, d^r, p, e are regular and S-related to e (cf. [1], [8]).

An easy consequence of this assertion and Corollary 2. to Proposition 3. is the following assertion.

Proposition 4. $(p^*)^+ = p, \quad (e^*)^+ = e, \quad ((d^a)^*)^+ = d^a, \quad ((d^r)^*)^+ = d^r.$

In other words, we have demonstrated that if we define t to be any of the topologies d^a, d^r, p, e, then t is the coarsest of all topologies primitive to t^* on $(\mathbb{R}, e)$.

With the aid of Corollary to Theorem 2., Proposition 5. reveals that in any case t is not the finest of all topologies primitive to t^*.

Proposition 5. *Let t be any of the topologies e, d^a, d^r, p. Then $t \subsetneq (t^*)^a$.*

PROOF. We already know that $t \subset (t^*)^a$, due to Corollary to Theorem 2.. To accomplish the proof it suffices to find a set $G \in e$ and a point $x \in \mathbb{R}$ such that $G \cup \{x\} \in t^* \setminus t$. Let S denote the lower density relevant to topology t^*.

1. Let $t = e$.

Set $G = (-1,1) \setminus (\{\pm\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\})$. Evidently, $G \in e$ and $\overline{G}^e = [-1,1]$. It follows from the properties of the category lower density S that $S((-1,1)) \subset S(\overline{G}^e) = S(G)$. Since $(-1,1) \in e \subset e^*$, $(-1,1) \subset S((-1,1)) \subset S(G)$. In prticular, $0 \in S(G)$. Consequently, $G \cup \{0\} \in e^*$. But obviously, $G \cup \{0\} \notin e$.

2. Let $t = d^a$ ($t = d^r$, respectively).

Choose an e-open set G such that $G \subset (0,1)$, $\lambda(G) < \frac{1}{2}$, $\overline{G}^e = [0,1]$ (such set G is sure to exit). Apparently, $(0,1) \subset S(G)$. Thus, $G \cup \{x\} \in t^*$ for any $x \in (0,1)$. It suffices to prove that there exists $x \in (0,1)$ with the property that $G \cup \{x\} \notin t$. Suppose that no such x exists in $(0,1)$, i.e. $G \cup \{x\} \in t$ for any $x \in (0,1)$. This also implies (since $t \subset d$) that $\forall x \in (0,1)$ we have $\lim_{h\to 0+} \frac{1}{2h}\lambda(G \cap (x-h,x+h)) = 1$. From this we can easily get that $(\forall x \in (0,1))(\exists h_0 > 0)(\forall 0 < h < h_0) : \frac{1}{2h}\lambda(G \cap (x-h,x+h)) > \frac{3}{4}$ and so $\lambda(G) > \lambda([0,1] \setminus G)$. This is contrary to the choice of the set G, for $\lambda(G) < \frac{1}{2}$.

3. Let $t = p$.

First, choose for $k = 1,2,\ldots$ e-open sets G_k such that

(i) $G_k \subset (\frac{1}{2^{k+1}}, \frac{1}{2^k})$

(ii) $\overline{G_k}^e = [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$

(iii) $2^k \cdot \lambda(G_k) < \frac{1}{k}$.

Note that such a choice is possible. Now define the set G by $G = (-1,0) \cup \bigcup_{k=1}^{\infty} G_k$. Obviously, $G \in e$ and $G \cap (\frac{1}{2^{k+1}}, \frac{1}{2^k}) = G_k$ for $k = 1,2,\ldots$. Moreover $(-1,\frac{1}{2}) \subset \overline{G}^e$ and hence $0 \in S(G)$.

To finish the proof of Proposition 5. it suffices to show that 0 is not a p-interior point of the set $G \cup \{0\}$. Suppose $G \cup \{0\}$ is p-open. Hence (cf. [3]), there exist $\varepsilon > 0$ and $\delta > 0$ such that for each $x \in (0,\delta)$ there exists an open interval I with the length at least $\varepsilon \cdot x$, for which

$$I \subset G \cap (x - \tfrac{1}{2}x, x). \tag{2}$$

Now choose an arbitrary positive integer k_0 such that $\frac{1}{2^{k_0}} < \delta$ and for integers $k > k_0$ define points $x_k = \frac{1}{2^k}$. By (2), for every such x_k there exists an open interval—denote it by J_k— not shorter than $\varepsilon \cdot x_k$ such that $J_k \subset G \cap (x_k - \frac{1}{2}x_k, x_k) = G \cap (\frac{1}{2^{k+1}}, \frac{1}{2^k}) = G_k$. Hence, for every integer $k > k_0$, $\lambda(G_k) \geq \lambda(J_k) \geq \varepsilon \cdot x_k = \varepsilon \cdot \frac{1}{2^k}$. Thus, for every integer $k > k_0$, $2^k \cdot \lambda(G_k) \geq \varepsilon$, which is contrary to condition (iii) for the choice of G_k.

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