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ALGEBRA GENERATED BY A.E. CONTINUOUS DERIVATIVES OF INTERVAL FUNCTIONS

Abstract

In this paper it is proved that each a.e. continuous Baire one function $u : \mathbb{R}^m \to \mathbb{R}$ can be written as $u = f \cdot g + h$, where f, g and h are a.e. continuous derivatives (with respect to the ordinary differentiation basis).

In 1982 D. Preiss proved the following theorem [7].

Theorem 1 *Whenever $u : \mathbb{R} \to \mathbb{R}$ is a function of the first class there are derivatives $f, g, h : \mathbb{R} \to \mathbb{R}$ such that $u = f \cdot g + h$. Moreover one can find such a representation that g is bounded and h is Lebesgue and in case u is bounded such that f and h are also bounded.*

Generalizations of this theorem for derivatives of interval functions (with respect to the ordinary differentiation basis) can be found in [1] and [4]. Another generalization of Preiss's theorem was published in 1990 [3].

Theorem 2 *Whenever $u : \mathbb{R} \to \mathbb{R}$ is a function of the first class which is continuous almost everywhere there are derivatives f, g and h which are continuous almost everywhere such that g is bounded and $u = f \cdot g + h$ and in case u is bounded such that f and h are also bounded.*

In this paper I prove an analogous theorem for derivatives of interval functions (with respect to the ordinary differentiation basis). The method of the proof is a modification of that of [3].

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First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. Throughout this article m is a fixed positive integer. The word function means mapping from $\mathbb{R}^m$ into $\mathbb{R}$ unless otherwise explicitly stated. The words measure, almost everywhere (a.e.), summable etc. refer to the Lebesgue measure and integral in $\mathbb{R}^m$. We denote by $a \vee b$ ($a \wedge b$) larger (smaller) of the real numbers a and b. The Euclidean metric in $\mathbb{R}^m$ will be denoted by ϱ. For every set $A \subset \mathbb{R}^m$, let $diam\, A$ be its diameter (i.e. $diam\, A = \sup\{\varrho(x,y) : x, y \in A\}$), χ_A its characteristic function and $|A|$ its outer Lebesgue measure. The symbol $\int_A f$ will always mean the Lebesgue integral. We say that f is a Baire one function, if it is the pointwise limit of a sequence of continuous functions. For any function f we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbb{R}^m\}$ (f needn't be bounded), and we denote by $\mathcal{D}(f)$ the set of points of discontinuity of f.

The word *interval* (*cube*) will always mean non-degenerate compact interval (cube) in $\mathbb{R}^m$, i.e. the Cartesian product of m non-degenerate compact intervals (compact intervals of equal length) in $\mathbb{R}$. We denote by Γ the family of all intervals.

For each interval $I = [a_1, b_1] \times \ldots \times [a_m, b_m]$, we set

$$I^\circ = [a_1, b_1) \times \ldots \times [a_m, b_m).$$

Let $n \in \mathbb{N}$. We say that I is a *basic cube of order* n, if

$$I = \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \ldots \times \left[\frac{k_m}{2^n}, \frac{k_m+1}{2^n}\right]$$

for some $k_1, \ldots, k_m \in \mathbb{Z}$. The family of all basic cubes of order n will be denoted by Γ_n. Elements of $\bigcup_{n=1}^{\infty} \Gamma_n$ will be called simply *basic cubes.*

Remark. Observe that for any two basic cubes I and J, either I and J do not overlap (i.e. $I \cap J \notin \Gamma$), or $I \subset J$, or $J \subset I$.

The following lemma is a slightly modified version of Lemma 2.1 of [5].

Lemma 3 *Let $A \subset \mathbb{R}^m$ be closed and let v be a function such that $\mathcal{D}(v) \subset A$. Then there exists a family $\mathcal{J}$ of non-overlapping basic cubes such that the following conditions are satisfied:*

i) $\bigcup \mathcal{J} = \mathbb{R}^m \setminus A$,

ii) each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}$,

iii) $diam\, J \leq \varrho(A, J)$ *for each* $J \in \mathcal{J}$, *iv) for each $\tau > 0$, there exists an $\eta > 0$ such that*

$$\|v \cdot \chi_J\| \leq \tau \cdot \varrho(A, J) \cdot |J|^{-1/m}$$

whenever $J \in \mathcal{J}$ and $\varrho(A, J) < \eta$.

PROOF. Let $\mathcal{I}$ be a family of basic cubes such that $\bigcup \mathcal{I} = \mathbb{R}^m \setminus A$ and each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{I}$ [5, Lemma 2.1]. For each $I \in \mathcal{I}$, since I is compact and $v|I$ is continuous, $C_I = \|v \cdot \chi_I\| < \infty$. Write I as the union $I = \bigcup_{i=1}^{k_I} J_{I,i}$ of non-overlapping basic cubes of diameter less than

$$\varrho(A, I) \wedge \frac{[\varrho(A, I)]^2}{C_I + 1}$$

and define $\mathcal{J} = \{J_{I,i} : \ I \in \mathcal{I}, \ i \in \{1, \dots, k_I\}\}$. Then clearly the requirements of the lemma are satisfied. □

By an *interval function* we will mean a mapping from Γ into $\mathbb{R}$.

We say that intervals I, J are *contiguous*, if they do not overlap and $I \cup J$ is an interval. We say that an interval function F is *additive*, if $F(I \cup J) = F(I) + F(J)$ whenever I and J are contiguous intervals.

We say that a sequence of intervals $\{I_n : \ n \in \mathbb{N}\}$ is

- *s-convergent* to a point $x \in \mathbb{R}^m$, if

  i) $x \in \bigcap_{n=1}^{\infty} I_n$,

  ii) $\lim_{n \to \infty} \mathit{diam}\, I_n = 0$.

- *o-convergent* to a point $x \in \mathbb{R}^m$, if the conditions i) and ii) above are fulfilled and moreover,

  iii) $\limsup_{n \to \infty} \dfrac{(\mathit{diam}\, I_n)^m}{|I_n|} < \infty$.

We will write $I_n \overset{s}{\Rightarrow} x$ and $I_n \overset{o}{\Rightarrow} x$, respectively. (Cf. e.g. [5].)

Let F be an arbitrary interval function and $x \in \mathbb{R}^m$. We define

$$s\text{-}\limsup_{I \Rightarrow x} F(I) = \sup \left\{ \limsup_{n \to \infty} F(I_n) : \ I_n \overset{s}{\Rightarrow} x \right\}.$$

In similar way we define $o\text{-}\limsup_{I \Rightarrow x} F(I)$, $s\text{-}\liminf_{I \Rightarrow x} F(I)$ etc.

We say that function f is an *o-derivative*, if there exists an additive interval function F (called the *primitive* of f) such that

$$\underset{I \Rightarrow x}{o\text{-}\lim} \frac{F(I)}{|I|} = f(x)$$

holds for each $x \in \mathbb{R}^m$. Recall that o-derivatives are Baire one functions (cf. [3, Lemma 2.1, p. 151] and [5, Lemma 3.1]).

We say that $x \in \mathbb{R}^m$ is an *o-Lebesgue point* of function f, if f is locally summable at x and

$$\underset{I \Rightarrow x}{o\text{-}\lim} \frac{\int_I |f - f(x)|}{|I|} = 0.$$

We say that f is an *o-Lebesgue function*, if each $x \in \mathbb{R}^m$ is an o-Lebesgue point of f.

Paper [5] introduced the notion of an *o-point* and the notion of an *o-integral* which is proper for integrating of o-derivatives. We will say that a function f is *integrable on an interval* I, if the o-integral of f on I exists. We will say that f is *locally integrable at a point* $x \in \mathbb{R}^m$, if there exists an $\eta > 0$ such that f is integrable on every interval $I \ni x$ of diameter less than η. The value of the o-integral of function f on an interval I will be denote by $\mathcal{S}_o(f, I)$.

We will say that $x \in \mathbb{R}^m$ is an *o-point* of the function f, if f is locally integrable at x and

$$\underset{I \Rightarrow x}{s\text{-}\lim} \frac{\mathcal{S}_o(f,I) - f(x) \cdot |I|}{(diam\, I)^m} = 0.$$

Remark. In the above definition we have the s-limit, which is much simpler than the o-limit, since we can express it in a Cauchy-like manner.

The notion of an o-point is a local characterization of an o-derivative.

Theorem 4 *Let f be an arbitrary function. Then f is an o-derivative if and only if each $x \in \mathbb{R}^m$ is an o-point of f.* [5, Theorem 5.4]

Remark. It is easy to prove the following assertions:

- If a function f is locally summable and continuous at $x \in \mathbb{R}^m$, then x is an o-Lebesgue point of f.
- If $x \in \mathbb{R}^m$ is an o-Lebesgue point of a function f, then it is an o-point.

The following theorems are proved in [5].

Theorem 5 *Assume that $A \subset \mathbb{R}^m$ is closed, $\mathcal{J} = \{J_n : n \in \mathbb{N}\}$ is a family of non-overlapping cubes and $\{f_n : n \in \mathbb{N}\}$ is a family of summable functions such that the following conditions are satisfied:*

i) each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}$,

ii) $|J_n| \leq [\varrho(A, J_n)]^m$, $n \in \mathbb{N}$,

iii) $f_n(x) = 0$, if $x \notin J_n$, $n \in \mathbb{N}$,

iv) for each $x \in A$ and each $\tau > 0$, there exists an $\eta(x, \tau) > 0$ such that for each $n \in \mathbb{N}$, if $\varrho(x, J_n) < \eta(x, \tau)$, then

a) $\left|\int_{J_n} f_n\right| \leq \tau \cdot |J_n|$, and

b) $\sup\left\{\left|\int_K f_n\right| : K \in \Gamma\right\} \leq \tau \cdot \varrho(x, J_n) \cdot |J_n|^{1-1/m}$.

Put $f = \sum_{n=1}^{\infty} f_n$. Then for each interval I, f is integrable on I, the series $\sum_{n=1}^{\infty} \int_I f_n$ is absolutely convergent, its sum equals $S_o(f, I)$ and moreover, each $x \in A$ is an o-point of f. [5, Theorem 6.1]

Theorem 6 *Whenever $A \subset \mathbb{R}^m$ is a closed set of measure zero and u is a Baire one function there exists an o-Lebesgue function φ such that $\mathcal{D}(\varphi) \subset A$ (so φ is continuous a.e.) and $\varphi(x) = u(x)$ for $x \in A$. Moreover one can choose φ such that $\|\varphi\| \leq \|u \cdot \chi_A\|$. [5, Corollary 7.9]*

Theorem 7 *Assume that the functions f_1, f_2 are such that for all $y \in \mathbb{R}^{m-1}$, $(0, y)$ is an o-point for both f_1 and f_2, and $f_1(0, y) = f_2(0, y)$. For $t \in \mathbb{R}$ and $y \in \mathbb{R}^{m-1}$ let*

$$h(t, y) = \begin{cases} f_1(t, y) & \text{if } t \geq 0, \\ f_2(t, y) & \text{if } t < 0. \end{cases}$$

Then for each $y \in \mathbb{R}^{m-1}$, $(0, y)$ is an o-point of h. [5, Theorem 7.1]

The proof of the following lemma is a repetition of arguments used to prove an analogous result in [3].

Lemma 8 *Whenever u is an a.e. continuous Baire one function there exist a.e. continuous Baire one functions $u_1, u_2, \ldots$ such that*

i) the sets $\mathcal{D}(u_1), \mathcal{D}(u_2), \ldots$ are closed,

ii) $\|u_k\| < 2^{-k}$ for $k \geq 2$,

iii) $u = \sum_{k=1}^{\infty} u_k$.

Lemma 9 *Assume that $I = [a_1, b_1] \times \ldots \times [a_m, b_m]$ is a cube, the functions ψ, $\tilde{f}$, $\tilde{g}$, $\bar{f}$ and $\bar{g}$ are summable on I and $\|\psi \cdot \chi_I\| = C < \infty$. Then there exist continuous functions f and g such that*

i) $f(x) = g(x) = 0$, *if* $x \notin I$,

ii) $\int_I f = \int_I g = \int_I (\psi - f \cdot g) = \int_I (g \cdot \tilde{f}) = \int_I (f \cdot \tilde{g}) = \int_I (g \cdot \bar{f}) = \int_I (f \cdot \bar{g}) = 0,$

iii) $\|f\| \leq 25 \cdot 2^m \cdot \left(\sqrt{C} \vee C\right)$, $\|g\| \leq \sqrt{C} \wedge 1$.

PROOF. For $i \in \{1, \ldots, 5\}$, let e_i be the function defined by

$$e_i(x_1, \ldots, x_m) = \sin \frac{2\pi i(x_1 - a_1)}{b_1 - a_1} \cdot \ldots \cdot \sin \frac{2\pi i(x_m - a_m)}{b_m - a_m}$$

for $(x_1, \ldots, x_m) \in \mathbb{R}^m$. The following system of equations

$$\begin{cases} z_1 \cdot \int_I (e_1 \cdot \tilde{f}) & + \ldots + & z_5 \cdot \int_I (e_5 \cdot \tilde{f}) & = 0 \\ z_1 \cdot \int_I (e_1 \cdot \tilde{g}) & + \ldots + & z_5 \cdot \int_I (e_5 \cdot \tilde{g}) & = 0 \\ z_1 \cdot \int_I (e_1 \cdot \bar{f}) & + \ldots + & z_5 \cdot \int_I (e_5 \cdot \bar{f}) & = 0 \\ z_1 \cdot \int_I (e_1 \cdot \bar{g}) & + \ldots + & z_5 \cdot \int_I (e_5 \cdot \bar{g}) & = 0 \end{cases}$$

is linear, homogeneous and the number of unknown quantities exceeds the number of equations, so it has a non-zero solution, say $\beta_1, \ldots, \beta_5$. Set

$$\gamma = \sqrt{\frac{2^m \cdot \left|\int_I \psi\right|}{(\beta_1^2 + \ldots + \beta_5^2) \cdot |I|}}$$

and $\alpha_i = \gamma \cdot \beta_i$ ($i \in \{1, \ldots, 5\}$). Let $e = \alpha_1 \cdot e_1 + \ldots + \alpha_5 \cdot e_5$. Then for $i \in \{1, \ldots, 5\}$,

$$\alpha_i^2 \cdot |I|/2^m \leq (\alpha_1^2 + \ldots + \alpha_5^2) \cdot |I|/2^m = \left|\int_I \psi\right| \leq C \cdot |I|,$$

so $|\alpha_i| \leq 2^{m/2} \cdot \sqrt{C}$ and $\|e\| \leq 5 \cdot 2^{m/2} \cdot \sqrt{C}$.

Define functions f and g by

$$f(x) = 5 \cdot 2^{m/2} \cdot \left(\sqrt{C} \vee 1\right) \cdot e(x) \cdot \chi_I(x)$$

and

$$g(x) = \frac{e(x) \cdot sgn \int_I \psi}{5 \cdot 2^{m/2} \cdot \left(\sqrt{C} \vee 1\right)} \cdot \chi_I(x).$$

It is easy to prove that conditions i)–iii) are satisfied. □

Theorem 10 *Whenever $u : \mathbb{R}^m \to \mathbb{R}$ is an a.e. continuous Baire one function there exist a.e. continuous o-derivatives f, g and h such that g is bounded and $u = f \cdot g + h$, and in case u is bounded such that f and h are also bounded.*

PROOF. First use Lemma 8 to find a.e. continuous Baire one functions $u_1, u_2, \ldots$ such that the sets $\mathcal{D}(u_1), \mathcal{D}(u_2), \ldots$ are closed, $\|u_k\| < 2^{-k}$ for $k \geq 2$ and $u = \sum_{k=1}^{\infty} u_k$. Put $A_0 = \emptyset$ and $\mathcal{J}_0 = \Gamma_1$. For $k \in \mathbb{N}$, we define functions f_k and g_k by induction as follows.

Let $A_k = A_{k-1} \cup \mathcal{D}(u_k)$. By Theorem 6, there exists an o-Lebesgue function φ_k such that

- $\mathcal{D}(\varphi_k) \subset A_k$,
- $\varphi_k(x) = u_k(x)$ for $x \in A_k$, and
- $\|\varphi_k\| \leq \|u_k \cdot \chi_{A_k}\|$.

Set $\psi_k = u_k - \varphi_k$. Use Lemma 3 with $A = A_k$ and $v = v_k = |\psi_k| \vee \sqrt{|\psi_k|}$ and find a family $\mathcal{J}_k = \{J_{k,n} : n \in \mathbb{N}\}$ of non-overlapping basic cubes such that

- $\bigcup \mathcal{J}_k = \mathbb{R}^m \setminus A_k$,
- each $x \notin A_k$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}_k$,
- *diam* $J_{k,n} \leq \varrho(A_k, J_{k,n})$ for each $n \in \mathbb{N}$,
- for each $\tau > 0$, there exists an $\eta > 0$ such that if $n \in \mathbb{N}$ and $\varrho(A_k, J_{k,n}) < \eta$, then
$$\|v_k \cdot \chi_{J_{k,n}}\| \leq \tau \cdot \varrho(A_k, J_{k,n}) \cdot |J_{k,n}|^{-1/m}.$$

We may also assume that $\mathcal{J}_k$ is a refinement of $\mathcal{J}_{k-1}$, i.e. each element of $\mathcal{J}_k$ is contained in one of elements of $\mathcal{J}_{k-1}$ (cf. Remark on p. 147). Put

- $\tilde{f}_k = \tilde{g}_k = \bar{f}_k = \bar{g}_k = 0$, if $k = 1$,
- $\tilde{f}_k = f_1$, $\tilde{g}_k = g_1$, $\bar{f}_k = f_2 + \ldots + f_{k-1}$, $\bar{g}_k = g_2 + \ldots + g_{k-1}$, if $k > 1$.

For each $n \in \mathbb{N}$, apply Lemma 9 with $I = J_{k,n}$, $\psi = \psi_k$, $\tilde{f} = \tilde{f}_k$, $\tilde{g} = \tilde{g}_k$, $\bar{f} = \bar{f}_k$ and $\bar{g} = \bar{g}_k$, and find continuous functions $f_{k,n}$, $g_{k,n}$ such that, setting $C_{k,n} = \|\psi_k \cdot \chi_{J_{k,n}}\|$, we have

- $f_{k,n}(x) = g_{k,n}(x) = 0$, if $x \notin J_{k,n}$,
- $$\int_{J_{k,n}} f_{k,n} = \int_{J_{k,n}} g_{k,n} = \int_{J_{k,n}} (\psi_k - f_{k,n} \cdot g_{k,n}) = \int_{J_{k,n}} (g_{k,n} \cdot \tilde{f}_k) = \int_{J_{k,n}} (f_{k,n} \cdot \tilde{g}_k) = \int_{J_{k,n}} (g_{k,n} \cdot \bar{f}_k) = \int_{J_{k,n}} (f_{k,n} \cdot \bar{g}_k) = 0,$$

- $\|f_{k,n}\| \leq 25 \cdot 2^m \cdot \left(\sqrt{C_{k,n}} \vee C_{k,n}\right)$, $\|g_{k,n}\| \leq \sqrt{C_{k,n}} \wedge 1$,

and set $h_{k,n} = \psi_k \cdot \chi_{J^\circ_{k,n}} - f_{k,n} \cdot g_{k,n}$.

Let

$$f_k = \sum_{n=1}^{\infty} f_{k,n}, \quad g_k = \sum_{n=1}^{\infty} g_{k,n} \text{ and } h_k = \varphi_k + \sum_{n=1}^{\infty} h_{k,n}.$$

Observe that $\|g_1\| \leq 1$ and for $k \geq 2$,

$$\begin{aligned} \|f_k\| &\leq 25 \cdot 2^m \cdot \sup\left\{\sqrt{C_{k,n}} \vee C_{k,n} : n \in \mathbb{N}\right\} \\ &\leq 25 \cdot 2^m \cdot \left(\sqrt{\|\psi_k\|} \vee \|\psi_k\|\right) \leq 2^{m+5+(1-k)/2} \end{aligned}$$

and $\|g_k\| \leq 2^{(1-k)/2}$. Hence for $k \geq 2$,

$$\begin{gathered} \|h_k\| \leq \|\varphi_k\| + \|\psi_k\| + \|f_k\| \cdot \|g_k\| \leq 2^{m+7-k}, \\ \|f_k \cdot \tilde{g}_k\| \leq 2^{m+6-k/2}, \\ \|g_k \cdot \bar{f}_k\| \leq 2^{m+5} \cdot \left(\sqrt{2}+1\right) \cdot \|g_k\| \leq 2^{m+7-k/2}, \\ \|f_k \cdot \bar{g}_k\| \leq 2^{m+7-k/2}. \end{gathered}$$

Consequently all of the series below converge uniformly and

$$\begin{aligned} u \;=\; & \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty}(f_k \cdot g_k + h_k) = \sum_{k=1}^{\infty} f_k \cdot \sum_{k=1}^{\infty} g_k \\ & - f_1 \cdot \sum_{k=2}^{\infty} g_k - \sum_{k=2}^{\infty}(f_k \cdot g_1) - \sum_{k=2}^{\infty}(f_k \cdot \bar{g}_k) - \sum_{k=2}^{\infty}(g_k \cdot \bar{f}_k) + \sum_{k=1}^{\infty} h_k. \end{aligned}$$

Set $f = \sum_{k=1}^{\infty} f_k$, $g = \sum_{k=1}^{\infty} g_k$ and

$$h = -f_1 \cdot \sum_{k=2}^{\infty} g_k - \sum_{k=2}^{\infty}(f_k \cdot g_1) - \sum_{k=2}^{\infty}(f_k \cdot \bar{g}_k) - \sum_{k=2}^{\infty}(g_k \cdot \bar{f}_k) + \sum_{k=1}^{\infty} h_k.$$

Then $u = f \cdot g + h$ and clearly f, g and h are continuous on $\mathbb{R}^m \setminus \bigcup_{k=1}^{\infty} A_k$. So they are a.e. continuous. For all $k \in \mathbb{N}$, use Theorem 5 with $A = A_k$, $\mathcal{J} = \mathcal{J}_k$ and families of the functions $\{f_{k,n} : n \in \mathbb{N}\}$, $\{g_{k,n} : n \in \mathbb{N}\}$, $\{h_{k,n} : n \in \mathbb{N}\}$, $\{f_{k,n} \cdot \tilde{g}_k : n \in \mathbb{N}\}$, $\{f_{k,n} \cdot \bar{g}_k : n \in \mathbb{N}\}$ and $\{g_{k,n} \cdot \bar{f}_k : n \in \mathbb{N}\}$. We get that each $x \in A_k$ is an o-point of functions f_k, g_k, h_k, and, if $k \geq 2$, of $f_k \cdot g_1$, $f_k \cdot \bar{g}_k$ and $g_k \cdot \bar{f}_k$. On the other hand, observe that each $x \notin A_k$ is a point of continuity of these functions and consequently also a o-point.

Since the limit of a uniformly convergent series of o-derivatives is again an o-derivative, we need only show that $f_1 \cdot \sum_{k=2}^{\infty} g_k$ is an o-derivative to complete the proof. We will use once more Theorem 5 with $A = A_1$, $\mathcal{J} = \mathcal{J}_1$ and the family of functions $\{f_{1,n} \cdot \sum_{k=2}^{\infty} g_k : n \in \mathbb{N}\}$. Since the product of a continuous function with a bounded o-derivative is again an o-derivative (cf. [2], [5] or [6]), for each $n \in \mathbb{N}$, $f_{1,n} \cdot \sum_{k=2}^{\infty} g_k$ is an o-derivative. The other conditions are also fulfilled. So each $x \in A_1$ is also an o-point of $f_1 \cdot \sum_{k=2}^{\infty} g_k$. On the other hand, by Theorem 7, each $x \notin A_1$ is an o-point of this function. □

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