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ON LIMITS WITHOUT EPSILONS

Abstract

In a recent paper "Limits without epsilons" by Darwin E. Peek, the concept of the convergence of real sequences has been completely characterized by six properties of a relation on sequences (determining Cauchy sequences). With the exception of the third property, called the "Squeezing Theorem", the characterization is shown to be minimal in the sense that each of the other five conditions is necessary to preserve the characterization. We answer in the positive the question whether also the remaining property is necessary and hence independent of the other five. Our construction is based on the properties of vector $\mathcal{L}_0^*$-groups.

1. INTRODUCTION

$\mathcal{L}_0^*$ As a rule, $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ will denote the natural numbers, integers and real numbers, respectively. The set of all strictly monotone maps of $\mathbb{N}$ into $\mathbb{N}$ is denoted by MON.

Let $A \subset \mathbb{R}^{\mathbb{N}}$ and let "$\equiv$" be a relation on $\mathbb{R}^{\mathbb{N}}$. Assume that the following conditions are satisfied:

A1. If $X = \langle x_n \rangle \in A$ and $a \in \mathbb{R}$, then $aX = \langle ax_n \rangle \in A$;

A2. If $X, Y, V, W, X+V, Y+W \in A$, $X \equiv Y$ and $V \equiv W$, then $Y + W \equiv X + V$;

A3. If $X, Y \in A, V \in \mathbb{R}^{\mathbb{N}}, X \equiv Y$ and $x_n \leq v_n \leq y_n, n \in \mathbb{N}$, then $V \in A$ and $V \equiv X$;

A4. If $X \in A$ and U, V are subsequences of X, then $U, V \in A$ and $U \equiv V$;

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A5. $\langle(-1)^n\rangle \notin A$;

A6. If $X \notin A$ and X is bounded, then X has two subsequences $U, V \in A$ such that $U \not\equiv V$.

Then $(A, \equiv)$ is called a convergence system (on $\mathbb{R}^{\mathrm{N}}$).

Let X be a nonempty set. For $S = \langle S(n)\rangle \in X^{\mathrm{N}}$ and $s \in$ MON, $S \circ s = \langle S(s(n))\rangle$ denotes the corresponding subsequence of S. The constant sequence generated by $x \in X$ is denoted by $\langle x\rangle$. We say that $L \subset X^{\mathrm{N}} \times X$ is an $\mathcal{L}$-convergence if $(\langle x\rangle, x) \in L$ for all $x \in X$ and if $(S, x) \in L$ implies $(S \circ s, x) \in L$ for all $s \in$ MON. If $(S, x) \in L$, then we say that S L-converges to x. We speak of an $\mathcal{L}_0$-convergence if each $S \in X^{\mathrm{N}}$ L-converges to at most one limit and we speak of an $\mathcal{L}^*$-convergence if L satisfies the Urysohn axiom (i.e. $(S, x) \in L$ whenever for each $s \in MON$ there exists $t \in MON$ such that $(S \circ s \circ t, x) \in L$). If all four conditions are satisfied, then L is said to be an $\mathcal{L}_0^*$-convergence. If X is a group and L is compatible with the group structure of X (i.e. $(\langle x_n - y_n\rangle, x - y) \in L$ whenever $(\langle x_n\rangle, x), (\langle y_n\rangle, y) \in L$), then we speak of an $\mathcal{L}$-group ($\mathcal{L}_0^*$-group if L is a compatible $\mathcal{L}_0^*$-convergence). If X is a vector space over a scalar field F and L is a group $\mathcal{L}_0^*$-convergence such that $(\langle a x_n\rangle, a x) \in L$ whenever $a \in F$ and $(\langle x_n\rangle, x) \in L$, then we speak of a vector $\mathcal{L}_0^*$-group. $\mathcal{L}_0^*$-rings, vector $\mathcal{L}_0^*$-spaces, etc., are defined analogously.

Let L be an $\mathcal{L}$-convergence on X. Define L^* as follows: $(S, x) \in L^*$ whenever for each $s \in MON$ there exists $t \in MON$ such that $(S \circ s \circ t, x) \in L$. It is known that L^* is an $\mathcal{L}^*$-convergence; it is called the Urysohn modification of L. Recall that the Urysohn modification preserves the uniqeness of limits and the compatibility with all algebraic operations mentioned before. Instead of the $\mathcal{L}$-notation the so called FLUSH-notation is used in the literature (cf. [5]).

Further information about $\mathcal{L}$-structures can be found in [1], [2], [3], [4], [5], [7], [8], [9], [10], [11], and the references therein.

Topological vector groups are dealt with in [6] and [13].

Denote M the usual convergence of sequences on the real line. In [12] it has been proved that the only convergence system (on $\mathbb{R}^{\mathrm{N}}$) consists of all M-convergent sequences equipped with an equivalence relation $\langle x_n\rangle \equiv \langle y_n\rangle$ iff $\lim(x_n - y_n) = 0$.

2. THE INDEPENDENCE OF THE SQUEEZING THEOREM

Let L be a vector group $\mathcal{L}_0^*$-convergence on $\mathbb{R}$. Let A_L be the set of all L-convergent sequences. Let $\langle x_n\rangle \equiv_L \langle y_n\rangle$ mean that L-$\lim(x_n - y_n) = 0$.

Proposition 2.1 *The following assertions are true.* (i) *"$\equiv_L$" is an equivalence relation.* (ii) *The system* $(A_L, \equiv_L)$ *satisfies conditions* A1, A2, A4, A5. (iii) *Let* $M \subset L$. *Then* $(A_L, \equiv_L)$ *satisfies condition* A6.

PROOF. Assertions (i) and (ii) are easy consequences of the properties of L.

(iii) Observe that a bounded sequence L-converges iff it M-converges. Thus A6 is trivial. This completes the proof. □

Clearly, to construct a system $(A_L, \equiv_L)$ satisfying all conditions Ai but A3, it suffices to construct a vector group $\mathcal{L}_0^*$-convergence L on R such that $M \subset L$ and some unbounded sequence L-converges to 0.

It is known that a group $\mathcal{L}$-convergence L is homogeneous (L-$\lim x_n = x$ iff L-$\lim(x_n - x) = 0$) and that the sequences L-converging to 0 form a special subgroup $\mathcal{N}(L)$ of the group of all sequences. Further, L has unique limits iff $\mathcal{N}(L)$ does not contain any constant sequence except $\langle 0 \rangle$. In this case the Urysohn modification L^* of L has unique limits, too.

Finally, to guarantee that L is a vector $\mathcal{L}$-group it suffices to assume that $\mathcal{N}(L)$ contains $\langle ax_n \rangle$ for each scalar a and each $\langle x_n \rangle$ in $\mathcal{N}(L)$ (cf. [7]).

Thus, it suffices to prove that the set $\mathcal{N}(M)$ of all sequences converging in the real line to 0 can be enlarged to a subgroup of R^N such that it contains the unbounded sequence $T = \langle 2^n \rangle$, it is closed with respect to subsequences and the multiplication by constants, and it does not contain any constant sequence except $\langle 0 \rangle$.

Lemma 2.2 *Let* $\sum_{i=1}^k a_i 2^{t_i(n)} + U(n) = r$ *for all* $n \in N$, *where* $k \in N$, $r, a_i \in R$, $a_i \neq 0$, $t_i \in MON$, $i = 1, \ldots, k$, $U \in R^N$, $\lim U(n) = 0$. *Then* $r = 0$.

PROOF. Contrariwise, suppose that $r \neq 0$.

CASE A. Assume that for all $n \in N$ we have $t_1(n) > t_2(n) > \ldots > t_k(n)$. Clearly, $\sum_{i=1}^k a_i (T \circ t_i)(n) = 2^{t_1(n)}(a_1 + \sum_{i=2}^k a_i 2^{t_i(n) - t_1(n)}), n \in N$. There are two possibilities.

1. There exists $t \in$ MON such that all sequences $t_2 \circ t - t_1 \circ t, \ldots, t_k \circ t - t_1 \circ t$ are strictly decreasing. Since $\lim(a_1 + \sum_{i=2}^k a_i 2^{t_i(t(n)) - t_1(t(n))}) = a_1$ and $\lim U(n) = 0$, the sequence $\sum_{i=1}^k \langle a_i \rangle T \circ t_i + U$ is unbounded, a contradiction.

2. There is $t \in$ MON such that for some $i \in \{2, \ldots, k\}$ the sequence $t_i \circ t - t_1 \circ t$ is constant. Then choose t in such a way that $\emptyset \neq K = \{i \in \{2, \ldots, k\}; t_i \circ t - t_1 \circ t$ is a constant sequence$\}$ and for each $i \in \{2, \ldots, k\} \setminus K$ the sequence $t_i \circ t - t_1 \circ t$ is strictly decreasing. If $a_1 + \sum_{i \in K} a_i 2^{t_i(t(n)) - t_1(t(n))} \neq 0$, then $\lim(a_1 + \sum_{i=1}^k a_i 2^{t_i(t(n)) - t_1(t(n))}) = a_1 + \sum_{i \in K} a_i 2^{t_i(t(n)) - t_1(t(n))}$. Since $\lim U(n) = 0$ the sequence $\sum_{i=1}^k \langle a_i \rangle T \circ t_i \circ t + U$ is unbounded, a contradiction. If $a_1 + \sum_{i \in K} a_i 2^{t_i(t(n)) - t_1(t(n))} = 0$, then $\langle r \rangle = \sum_{i \in \{2, \ldots, k\} \setminus K} \langle a_i \rangle T \circ t_i \circ t +$

$U, \emptyset \neq K \subset \{2, \dots, k\}, \{2, \dots, k\} \setminus K \neq \emptyset$. Repeating steps 1 and 2, we reduce the complexity of Case A or arrive to a contradiction with $r \neq 0$.

CASE B. By a suitable choice of $s \in$ MON, passing to subsequences $t_i \circ s$, permuting the index set $\{1, \dots, k\}$ and reducing the index set (if $t_i \circ s = t_j \circ s$), the equation $\langle r \rangle = \sum_{i=1}^{k} \langle a_i \rangle T \circ t_i + U$ can be modified so that it will satisfy the assumptions of CASE A. Now proceed as in CASE A.

This completes the proof. □

Corollary 2.3 *Let L be the smallest vector group $\mathcal{L}_0^*$-convergence on R such that $M \subset L$ and L-$\lim 2^n = 0$ (unique limits of L are guaranteed by Lemma 1.2). Then the system $(A_L, \equiv_L)$ satisfies conditions A1, A2, A4, A5, A6, but does not satisfy condition A3.*

This solves the problem posed in [12]. Independently, the same problem has been solved in [8]. Observe that the underlying set A of the system $(A, \equiv)$ constructed there is in a certain sense minimal; it is not closed with respect to the addition of sequences.

3. REMARKS

The sequential continuity of algebraic operations has been studied since the early stages of Topological Algebra (cf. [14]). Usually the convergence is understood as a relation between sequences and points. If the underlying set is equipped with an order, then it is natural to define the convergence in terms of the order. As already mentioned in Section 1, a homogeneous compatible sequential convergence on a set X equipped with some algebraic operations can be identified with a suitable substructure of X^N consisting of "small" sequences. Usually, four types of axioms are considered:

1. Some distinct sequences are, or are not, "small";
2. "Small" sequences are closed with respect to some pointwise algebraic operations in X^N;
3. Each "smaller" sequence is "small";
4. "Small" sequences determine an invariant and we are interested in the largest set of "small" sequences determining the same invariant (e.g. for $\mathcal{L}_0$-space the Urysohn axiom is determined by the topological adherence).

Along these lines the convergence of sequences on the real line can be characterized, e.g., as follows.

Consider a set $\mathcal{S} \subset R^N$ satisfying the following axioms:

S1. $\langle 1/n\rangle \in \mathcal{S}$, $\langle 1\rangle \notin \mathcal{S}$;

S2. $\mathcal{S}$ is a Z-module;

S3. Let $\langle x_n\rangle \in \mathcal{S}$. Then $\langle x'_n\rangle \in \mathcal{S}$ whenever either

(i) $\langle x'_n\rangle$ is a subsequence of $\langle x_n\rangle$;

or

(ii) $0 \leq x'_n \leq x_n$ for all $n \in N$;

S4. If each subsequence $\langle x'_n\rangle$ of $\langle x_n\rangle$ contains a subsequence $\langle x''_n\rangle \in \mathcal{S}$, then $\langle x_n\rangle \in \mathcal{S}$.

If $\langle x_n - x\rangle \in \mathcal{S}$, then we say that $\langle x_n\rangle$ $\mathcal{S}$-converges to x. Then $\mathcal{S}$ is exactly the set of all sequences converging on the real line to 0 and hence a sequence $\langle x_n\rangle$ $\mathcal{S}$-converges to x iff it does on the real line. (Hint. If a sequence $\langle x_n\rangle$ M-converges to 0, then each of its subsequences $\langle x'_n\rangle$ contains a subsequence $\langle x''_n\rangle$ such that $|x_n| < 1/n$ for all $n \in N$. It follows from S4 that $\langle x_n\rangle$ $\mathcal{S}$-converges to 0. The converse is trivial.)

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