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MEASURE SPACES AND DIVISION SPACES

Abstract

The paper constructs a division space from an arbitrary non-atomic measure space with a locally compact topology that is compatible with the measure, and defines two equivalent integrals.

Many non-atomic measure spaces are constructed geometrically from simple objects, with division space integrals produced from them, giving a great economy. For example, in Euclidean n-dimensional space we use n-dimensional rectangles with sides parallel to the co-ordinate axes to construct Lebesgue measure; rectangles alone are used for the gauge or Kurzweil-Henstock integral, with no need *a priori* of the measure of any more general set, see [7]. For measure we can use the variation or inner variation.

Recently Lee Peng-Yee asked me to construct a division space from a non-atomic measure space given axiomatically, or one for which the original construction from simple objects is lost or ignored.(A measure space is *non-atomic* if every point has measure zero.) By our construction using [3], more Lebesgue theory comes under generalized Riemann theory, usually with simpler proofs. Note that Zeev Schuss [13] proved directly that in one dimension the Lebesgue integral is included in the gauge integral. R .O. Davies found faults and gave a more general accurate proof in [3]. Details in [7], Theorem 3.4, pp. 37–38; [8], Theorem 0.1.1, pp. 3–5, are not as full as in [3], so that here I give the details for finite real or complex valued functions f Lebesgue integrable over a measurable set M of finite m-measure, with a topology T. For Banach space valued f see [8]. Complex valued measures m are by the Radon-Nikodym theorem replaced by non-negative finitely (or countably) additive measures m, the complexity being transferred to the integrand. With the Lebesgue integral F of f, Davies used open neighborhood $G(x)$ of points $x \in M$, the G depending on a given $\varepsilon > 0$, and proved that when $I_1, I_2, \ldots$ are *essentially disjoint (i.e.*

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$m(I_j \cap I_k) = 0, j \neq k)$ measurable subsets of M with $m(M \setminus \bigcup_j I_j) = 0$, and $x_1, x_2, \ldots$ are points in M satisfying $x_j \in I_j \subseteq G(x_j)(j = 1, 2, \ldots)$ then

$$\left|\sum_j f(x_j)m(I_j) - F\right| < \varepsilon. \tag{1}$$

Using this, the *Davies integral of f over a measurable $N \subseteq M$* is defined as a number F such that for each $\varepsilon > 0$ an open neighborhood function $G : M \to T$ exists with $x \in G(x)$ $(all\, x \in M)$, and if (I_j) is a sequence of essentially disjoint measurable subsets with union N less a set of m-measure zero, and $(x_j) \subseteq M$ a sequence with $x_j \in I_j \subseteq G(x_j)$ $(all\ j)$ then (1) is true.

Given $\varepsilon > 0$, a valid definition needs some (I_j, x_j) satisfying the conditions, with F uniquely defined. For example, if $G \cap N$ is m-measurable for every open set G, if, omitting empty sets,

$$\begin{aligned} J_1 &= G(x_1) \cap N, J_2 = G(x_2) \cap N \setminus G(x_1), \\ J_3 &= G(x_3) \cap N \setminus (G(x_1) \cup G(x_2)), \ldots, \end{aligned} \tag{2}$$

we have disjoint measurable $J_j \subseteq G(x_j)$ with union $N \setminus Z$, $m(Z) = 0$. We later show F uniquely defined. We have a McShane cover of $N \setminus Z$ as $x_j \in J_j$ need not be true for all j. For a Davies cover of N we put $I_j = x_j \cup J_j$ $(j = 1, 2, \ldots)$, $m(x_j) = 0$, $m(I_j) = m(J_j)$, and (1) is satisfied with I_j replaced by the non-empty $J_j.Z$ is needed as in (1) the $G(x_j)$ need not cover all N.

If, given an arbitrary open cover $\mathcal{C}$ of N, an open neighborhood function G exists such that for each $x \in N$ a $C(x) \in \mathcal{C}$ has $G(x) \subseteq C(x)$, then by (1) we can say that *N is essentially covered* by the I_j, by the $G(x_j)$, and so by the $C(x_j)$, and that every open cover of N contains a countable essential subcover. Division space theory uses *finite* collections of (I, x) for easy proofs of various integral properties. Then in (1) the sum is finite, every open cover of N contains a finite essential subcover, and we can say that *N is essentially compact ; a priori* the set of m-measure zero varies with G. [4], [5] use *countable* collections of (I, x).

These preliminary remarks may suggest improvements of the rest of the paper, which begins with a McShane type fully decomposable division space from a non-atomic measure space of finite m-measure with a locally compact topology. The Alexandrov one-point compactification gives a compact topology T and an extra point ∞ that can be included in M; then M is a compact set relative to T. Let $G : M \to T$ be an open neighborhood function (so with $x \in G(x)$ for all $x \in M$). For a non-empty m-measurable $J \subseteq M$ and $x \in M$, the *generalized interval-point pair (J, x) is G-fine* if $J \subseteq G(x)$. A *division $\mathcal{D}$* of an m-measurable $N \subseteq M$, is a *finite* collection (J_j, x_j) $(1 \leq j \leq n)$ with disjoint J_j of union N, and $x_j \in M$. For example, use the compactness and

(2). The corresponding collection of J_j is called a *partition* $\mathcal{P}$ of N. $\mathcal{D}$ *is* G*-fine* if every $(J_j, x_j) \in \mathcal{D}$ is G-fine. A partition $\mathcal{P}_1$ of N *refines* a partition $\mathcal{P}_2$ of N if for each $J \in \mathcal{P}_1$, $J \subseteq I$ for some $I \in \mathcal{P}_2$. Let the finite valued $f : M \to \mathbb{R}$ (or $\mathbb{C}$). A number F *is the Davies- McShane integral of* f (*or* $f dm$) *over* N if, given $\varepsilon > 0$, an open neighborhood function $G : M \to T$ exists and, for all G-fine divisions $\mathcal{D}$ of N, $|(\mathcal{D}) \sum f(x)m(I) - F| < \varepsilon$. As in [8], pp. 40–43 we have a *division system for* N. For if $G_j : M \to T$ $(j = 1, 2)$ are two open neighborhood functions, $G_1 \wedge G_2$ is also, where $x \in G_1 \wedge G_2(x) = G_1(x) \cap G_2(x)$ (*all* $x \in M$). Hence we have a division system for N. If $G_1 = G$, G_2 the G for (1), $G_1 \wedge G_2$ satisfies (1), the collection of divisions is a subset of the arrangements just before (1), and if the Davies integral exists, so does the Davies-McShane integral with the same value. If $F = F_j$ using $G = G_j$ $(j = 1, 2)$ for the Davies-McShane integral, a $G_1 \wedge G_2$-fine division $\mathcal{D}$ of N satisfies

$$|(\mathcal{D}) \sum f(x)m(I) - F_j| < \varepsilon (j = 1, 2), |F_1 - F_2| < 2\varepsilon,$$

true for all $\varepsilon > 0$, $F_1 = F_2$, and the Davies-McShane integral is uniquely defined; therefore so is the Davies integral when it exists. If an open neighborhood function $G_x : M \to T$ exists, for each $x \in M$, and $G_3(x) = G_x(x)$ $(x \in M)$, then G_3 is an open neighborhood function, and the division system is *fully decomposable.* N and each non-empty measurable subset $N_1 \subset N$ are partial sets of N. For $N \setminus N_1$ is not empty, and so, reverting to the earlier open neighborhood function G, there are G-fine divisions $\mathcal{D}_1$ of N_1, $\mathcal{D}_2$ of $N \setminus N_1$, and $\mathcal{D}_1 \cup \mathcal{D}_2$ is a G-fine division of N, N_1 being the union of I from the $(I, x) \in \mathcal{D}_1$. As the G are given for all $x \in M$, not just for $x \in N$, the division space has the *restriction property.* For $N_1, ..., N_n$ partial sets of N, they are non-empty measurable sets, every non-empty $N_j \cap N_k$ is measurable and so a partial set, and the $N_1, ..., N_n$ are *co-partitional.* Thus we have a *division space*, previously called a *non-additive division space.* It is not always *additive* as the simple McShane example of [8], section 1.6, p. 54, and the example of [8], section 1.4, pp. 50–52, show. This is no barrier to proving that the Lebesgue integral exists when the Davies-McShane integral exists, as m is at least finitely additive.

Next we use two ideas, first, one common to all division systems. Generalized intervals I are used to construct partitions of N, points x associated with them being used to find which (I, x) are relevant in Riemann sums. If I is so connected with both x and y, (I, x) or (I, y) can be used in the division without altering the partition. The second idea is special to McShane-type division spaces. If $I \supseteq J$ are two generalized intervals and (I, x) is fine, so is (J, x). Given two divisions $\mathcal{D}_1$, $\mathcal{D}_2$ of N, a division $\mathcal{D}_3$ exists for which the corresponding partition refines the partitions from $\mathcal{D}_1$ and $\mathcal{D}_2$. If $(J, t) \in \mathcal{D}_3$ there are $(I_j, x_j) \in \mathcal{D}_j$ with $J \subseteq I_j$ $(j = 1, 2)$ and t can be x_1 or x_2. J runs through all non-empty intersections $I_1 \cap I_2$.

Theorem 1 *Given $\varepsilon > 0$, for $G : M \to T$ let every two G- fine divisions $\mathcal{D}_1$, $\mathcal{D}_2$ of N satisfy*

$$(3) \qquad |(\mathcal{D}_1)\sum f(x_1)m(I_1) - (\mathcal{D}_2)\sum f(x_2)m(I_2)| < \varepsilon.$$

Then for the corresponding $\mathcal{D}_3$, and $k = 1$ (f real valued), 2(f complex valued),

$$(4) \qquad (\mathcal{D}_3)\sum |f(x_1) - f(x_2)|m(J) < k\varepsilon.$$

PROOF. For real valued f, $(J,t) \in \mathcal{D}_3$, $J = I_1 \cap I_2$,and $f(t)$ the greater of $f(x_1)$, $f(x_2)$, we have a division $\mathcal{D}_4$, say. $\mathcal{D}_5$ is for $f(t)$ the smaller of $f(x_1)$, $f(x_2)$. As $\mathcal{D}_4$, $\mathcal{D}_5$ are like $\mathcal{D}_1$, $\mathcal{D}_2$, with the I_1, I_2 replaced by J, (4) follows from (3). For complex valued f, split into real and imaginary parts and use [8], Theorem 2.3.3(2.3.8), p. 77. □

Theorem 2 *$|f|\ dm$ is integrable.*

PROOF. In Theorem 1, as $m \geq 0$ is finitely additive and f is real or complex valued,

$$\left|(\mathcal{D}_1)\sum |f(x_1)|m(I_1) - (\mathcal{D}_2)\sum |f(x_2)|m(I_2)\right| =$$

$$\left|(\mathcal{D}_3)\sum \{|f(x_1)| - |f(x_2)|\}m(J)\right| \leq (\mathcal{D}_3)\sum \big||f(x_1)| - |f(x_2)|\big|m(J) \leq$$

$$(\mathcal{D}_3)\sum |f(x_1) - f(x_2)|m(J) < k\varepsilon,$$

a Cauchy-type convergence condition for integrability, proving the result. Taking real and imaginary parts, and $(|f|+f)/2$, $(|f|-f)/2$(f *real*) we need only have $f \geq 0$. □

Theorem 3 *For a number $q > 0, f : M \to \mathbb{R}_+$, and χ the indicator of the set $X(f \geq q)$ of points where $f \geq q$, if $f\ dm$ is Davies-McShane integrable over N, then $\chi\ dm$ is Davies-McShane integrable over N.*

PROOF. For arbitrary real numbers x_j , y_j $(j = 1, 2)$,

$$(5) \qquad \begin{gathered} \min(x_1, x_2) \leq x_j = (x_j - y_j) + y_j \leq |x_1 - y_1| + |x_2 - y_2| + y_j \ (j = 1, 2), \\ \min(x_1, x_2) - \min(y_1, y_2) \leq |x_1 - y_1| + |x_2 - y_2|. \end{gathered}$$

$$(6) \qquad \begin{gathered} \min(x_1, x_2) + \min(y_1, y_2) \leq x_j + y_j \ (j = 1, 2), \\ \min(x_1, x_2) + min(y_1, y_2) \leq \min(x_1 + y_1, x_2 + y_2). \end{gathered}$$

Result (5) gives, for F the indefinite integral of $f\ dm$ and

$$\Delta \equiv \min(fm, qm) - \min(F, qm), |\Delta| \leq |fm - F| + 0,$$

and the Davies-McShane variation of Δ is 0. $\min(F, qm)$ is finitely superadditive, $\min(F(I), qm(I)) \geq \min(F(I_1), qm(I_1)) + \min(F(I_2, qm(I_2))$ ($I = I_1 \cup I_2$, $I_1 \cap I_2$ *empty)* by (6) and the finite additivity of F and m. By refinements of partitions, sums of $\min(F, qm) \geq 0$ are monotone decreasing to a refinement limit, say L. This with the zero variation of Δ gives $\min(f, q)\ dm$ integrable by refinements plus Davies-McShane integration. To remove the refinements, given $\varepsilon > 0$, let $\mathcal{P}$ be a partition of N with n generalized intervals, such that

(7) $$L \leq (\mathcal{P}) \sum \min(F, qm) < L + \varepsilon/2.$$

By the zero variation of Δ, for each $I \in \mathcal{P}$ let $G_I : M \to T$ be an open neighborhood function such that every G_I-fine division $\mathcal{D}_I$ of I satisfies

(8) $$(\mathcal{D}_I) \sum |\Delta| < \varepsilon/(2n).$$

For each $x \in M$ let $G(x)$ be the intersection of $G_I(x)$ for the finite number of $I \in \mathcal{P}$, and let $\mathcal{D}$ be a G-fine division of N. Then the

$$(J \cap I, t)\ (I \in \mathcal{P},\ (J, t) \in \mathcal{D}, J \cap I \textit{ non-empty})$$

form a G-fine division of N. As m is finitely additive the split of J into the $J \cap I$ leaves the value of the Riemann sum unaltered. $\mathcal{D}$ becomes a union of divisions $\mathcal{D}_I$, so that (8) and (7) give $L - \varepsilon < (\mathcal{D}) \sum \min(f, q)m < L + \varepsilon$,the integrability of $\min(f, q)m$ to L without refinements. Similarly, for numbers $p, q, 0 \leq p < q$, and $f_{pq} \equiv \max(\min(f, q), p)$, $f_{pq}m$is Davies-McShane integrable. So is $(f_{pq} - p)m/(q - p)$.The multiplier of m is $0(f \leq p), 1(f \geq q)$,lies between 0 and 1 if $p < f < q$, and as $p \to q-$, is monotone decreasing to χ, the indicator of $X(f \geq q)$. The refinement division space is not decomposable but the Davies -McShane division space is fully decomposable, so that [8], Theorem 3.2.1, p. 126, applies and $\chi\ dm$ is Davies-McShane integrable. □

Measures are assumed non-negative and countably additive,with a third axiom to deal with Borel sets. Carathéodory [2], p. 239, axiom IV, and Saks [12], p. 43, (C_3), depend on a metric in the space and Bourbaki [1] generalized them. Thomson [14], p. 291, Definition 4 and further remarks, defined *compatibility with a topology*, and Henstock [6], pp. 74, 75, gave an equivalent axiom:-cl denoting closure, if there are disjoint open sets G_1 ,G_2 with $\mathrm{cl}\, X \subseteq G_1$, $\mathrm{cl}\, Y \subseteq G_2$, *then* $m(X) + m(Y) = m(X \cup Y)$. The Davies-McShane variation satisfies this; an open neighborhood function $G : M \to T$ exists with $G(t) \subseteq G_1(t \in \mathrm{cl}\, X), G(t) \subseteq G_2(t \in \mathrm{cl}\, Y)$,simply by replacing $G(t)$

by $G(t) \cap G_j$ for $j = 1, 2$ in the two cases. The Lebesgue integral of each closed set in M exists, and by [3] the Davies-McShane integral exists, so that for the indicator of $Y \subseteq M$,

$$(9) \qquad (D - McSh) \int_M \chi(Y;.)dm = m(Y)$$

for closed Y, and so for all Borel sets Y, and all m-measurable sets Y.

Theorem 4 *If f is m- measurable and fdm Davies-McShane integrable over N, it is Lebesgue integrable there.*

PROOF. For $f \geq 0$, and numbers $0 \leq p < q$, the set where $p \leq f < q, X(p \leq f < q) = X(f \geq p) \setminus X(f \geq q)$,and its indicator, times m, is Davies-McShane integrable over N, say to $Q(p, q)$. The indicator is also Lebesgue integrable as f is m-measurable, to $Q(p, q)$ by [3]. For $p = 0, 1/n, 2/n, \ldots$ and $q = p + 1/n$, the two sums

$$(10) \qquad \sum_{j=1}^{\infty} jQ(j/n, (j+1)/n)/n, \sum_{j=0}^{\infty}(j+1)Q(j/n, (j+1)/n)/n$$

lie below and above the Davies-McShane integral of fdm, with difference $\sum_{j=0}^{\infty} Q(j/n, (j+1)/n)/n = Q(0, +\infty)/n \to 0(n \to \infty)(0 \leq Q(0, +\infty) = m(N) \leq m(M) < +\infty)$.Thus both sums in (10) tend to the Davies-McShane integral of $f\ dm$. These sums are the Lebesgue way of integrating fdm, with an extension for unbounded non-negative functions, since $Q(p, q)$ is the value of the Lebesgue integral of the indicator of $X(p \leq f < q)$. □

If the Lebesgue integral of the m-measurable f (or $f\ dm$) exists with a finite value, so does the Davies integral, by [3]. If the Davies integral of $f\ dm$ exists, earlier remarks show that the Davies-McShane integral exists. By Theorem 4, if the Davies-McShane integral of $f\ dm$ exists with f m-measurable, the Lebesgue integral exists. Thus when f is m-measurable all three integrals of $f\ dm$ are equivalent. It is not proved that f is m-measurable when $f\ dm$ is Davies-McShane integrable.

It might be thought that T should be replaced by the *intrinsic topology* ([8], pp. 103–112). To find it we take m-measurable subsets N of M. Let $G : M \to T$ be an open neighborhood function with N_G* the set of t with $G(t) \cap N$ not empty. Then $N* \equiv \cap_G(N_G)* = \mathrm{cl}\, N$.For if $t \notin \mathrm{cl}\, N, t$ is in an open set G_4 with $G_4 \cap N$ empty. For $G(t) \subseteq G_4, t \notin (N_G)*$. Hence $N* \subseteq \mathrm{cl}\, N$. Conversely, if $t \in \mathrm{cl}\, N$, every $G(t) \cap N$ is not empty, and $N* = \mathrm{cl}\, N$. For all (I, t), $t \in M$. Thus the intrinsic topology is the empty set, M, and all complements $M \setminus N* = M \setminus \mathrm{cl}\, N \in T$. Conversely, if $G_5 \in T, G_5 \subseteq M$ and is m-measurable, and $\setminus G_5$ is a generalized interval. Hence T is the intrinsic topology.

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