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MONOTONICITY THEOREMS FOR SOME LOCAL SYSTEMS

Let A be a subset of the real line $\mathbb{R}$. Then $|A|^i$ denotes the inner Lebesgue measure of A and put

$$\underline{d}^i_-(A,x) = \liminf_{h\to 0^+} \frac{|A\cap(x-h,\, x)|^i}{h}$$

$$\underline{d}^i_+(A,x) = \liminf_{h\to 0^+} \frac{|A\cap(x,\, x+h)|^i}{h}$$

D. N. Sarkhel and A. K. De proved the following theorem [2, Lemma 2.3].

Theorem A *Let $A \subset [a,b]$ and $B = [a,b]\setminus A$. If*

(a) $a \in A$,

(b) $\underline{d}^i_+(A,x) > 0$ *for* $x \in A\setminus\{b\}$,

(c) $\underline{d}^i_-(B,x) > 0$ *for* $x \in B$.

Then $B = \emptyset$.

As a consequence, they established a monitonicity theorem [2, Theorem 4.3]. In this paper their theorem is generalized and a result equivalent to Theorem 55.13 in B. Thomson's book [3] is proved.

By a local system we mean a family $\mathbb{S} = \{\mathbb{S}(x);\ x \in \mathbb{R}\}$ of nonempty collections of subsets of the real line such that for every $x \in \mathbb{R}$

(i) $\{x\} \notin \mathbb{S}(x)$,

(ii) if $S \in \mathbb{S}(x)$, then $x \in S$,

(iii) if $S \in \mathbb{S}(x)$ and $S' \supset S$, then $S' \in \mathbb{S}(x)$,

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(iv) if $S \in \mathbb{S}(x)$ and $\delta > 0$, then $S \cap (x-\delta, x+\delta) \in \mathbb{S}(x)$.

If a local system $\mathbb{S}$ is bilateral (i.e. x is a bilateral accumulation point of any set from $\mathbb{S}(x)$), then we put

$$\mathbb{S}^-(x) = \{A;\ A \cup (x, \infty) \in \mathbb{S}(x)\}$$

$$\mathbb{S}^+(x) = \{A;\ A \cup (-\infty, x) \in \mathbb{S}(x)\}.$$

Clearly, $\mathbb{S}^-$ and $\mathbb{S}^+$ are local systems such that $S \cap [x, \infty) \in \mathbb{S}^+(x)$ when $S \in \mathbb{S}^+(x)$ and $\mathbb{S} \cap (-\infty, x] \in \mathbb{S}^-(x)$ when $S \in \mathbb{S}^-(x)$. Moreover, if a system $\mathbb{S}$ is filtering (i.e. $S_1 \cap S_2 \in \mathbb{S}(x)$ for any S_1, S_2 from $\mathbb{S}(x)$), then $\mathbb{S}(x) = \mathbb{S}^+(x) \cap \mathbb{S}^-(x)$ for every x.

Definition 1 *We say that a local system $\mathbb{S}$ fulfils condition (SD) if $\mathbb{S}$ is bilateral and for any closed interval $[a,b]$ and any sets $A \subset [a,b]$, $B = [a,b] \setminus A$, the following conditions*

(a) $a \in A$,

(b) $A \in \mathbb{S}^+(x)$ *for* $x \in A \setminus \{b\}$,

(c) $B \in \mathbb{S}^-(x)$ *for* $x \in B$

imply that $B = \emptyset$.

Notice that Theorem A guarantees that the local system $\mathbb{S}(x) = \{A;\ x \in A,\ \underline{d}^i_+(A,x) > 0 \text{ and } \underline{d}^i_-(A,x) > 0\}$ satisfies condition (SD).

We quote a few definitions from Thomson's book [3] which are necessary to formulate the announced theorems.

Let $\mathbb{S}$ be a local system, ψ - an interval function, C - a nonempty family of intervals, I - an interval and X - a subset of the real line. We say that C is a $(\mathbb{S})$-cover of X if, for each $x \in X$, there is $S \in \mathbb{S}(x)$ such that $[x,y] \in C$ for any $y \in S \setminus \{x\}$ ($[x,y]$ denotes the closed interval with endpoints x and y regardless of whether $x < y$ or $x > y$). We say that a subadditive nonnegative interval function ψ is $(\mathbb{S})$-continuous at x if, for every positive ε, there is a set $S \in \mathbb{S}(x)$ such that $\psi[x,y] < \varepsilon$ when $y \in S \setminus \{x\}$. Furthermore, put

$$\operatorname{Var}_I(\psi, C) = \sup\left\{\sum_{i=1}^{n} |\psi(I_i)|;\ I_i \in C,\ I_i \subset I,\ \operatorname{int} I_i \cap \operatorname{int} I_j = \emptyset \text{ for } i \neq j\right\}$$

$$V_I(\psi, \mathbb{S}, X) = \inf\{\operatorname{Var}_I(\psi, C);\ C \text{ is an } (S)\text{-cover of } X\},$$

$$V_I(\psi, \mathbb{S}) = V_I(\psi, \mathbb{S}, R).$$

Now we prove our lemma which corresponds to Lemma 38.6 from [3].

Lemma 1 *Let $\mathbb{S}$ be a local system satisfying condition (SD) and ψ a nonnegative subadditive interval function. If ψ is ($\mathbb{S}^-$)-continuous, then $\psi(I) \leq V_I(\psi, \mathbb{S}^+)$ for every interval I.*

PROOF. Let C be a ($\mathbb{S}^+$)-cover of the real line and let $[a,b]$ be an interval. Set $B = \{z \in (a,b];\ \psi[a,z] > \mathrm{Var}_{[a,z]}(\psi, C)\}$ and $A = [a,b] \setminus B$. Let ε be a positive number. Suppose that $x \in A \cap (a,b)$. Then there exist intervals $I_1, I_2, \ldots, I_n \in C$ included in $[a,x]$ for which $\psi[a,x] \leq \sum_{i=1}^{n} \psi(I_i) + \varepsilon$. Since C is n ($\mathbb{S}^+$)-cover, there is $S \in \mathbb{S}^+(x)$ such that $[x,y] \in C$ for $y \in S \setminus \{x\}$. Put $S_1 = S \cap [x,b)$. Then $S_1 \in \mathbb{S}^+(x)$ and for any $y \in S_1 \setminus \{x\}$ and we have

$$\psi[a,y] \leq \psi[a,x] + \psi[x,y] \leq \sum_{i=1}^{n} \psi(I_i) + \psi[x,y] + \varepsilon.$$

Thus $\psi[a,y] \leq \mathrm{Var}_{[a,y]}(\psi, C) + \varepsilon$ and by the arbitrariness of ε, $\psi[a,y] \leq \mathrm{Var}_{[a,y]}(\psi, C)$. Hence $y \in A$ and consequently, $S_1 \subset A$. This proves that $A \in \mathbb{S}^+(x)$.

In the case $x = a$, the condition $A \in \mathbb{S}^+(a)$ is evident.

Now, suppose that $x \in B$. Then $\psi[a,x] > \mathrm{Var}_{[a,x]}(\psi, C) + \varepsilon$ for some positive ε. Since ψ is ($\mathbb{S}^-$)-continuous, there exists $T \in \mathbb{S}^-(x)$ such that $\psi[z,x] < \varepsilon$ for $z \in T \setminus \{x\}$. Put $T_1 = T \cap (a,x]$. Then $T_1 \in \mathbb{S}^-(x)$ and for any $z \in T_1$, $\psi[a,z] \geq \psi[a,x] - \psi[z,x] > \mathrm{Var}_{[a,x]}(\psi, C) \geq \mathrm{Var}_{[a,z]}(\psi, C)$. Thus $z \in B$ and so, $B \in \mathbb{S}^-(x)$. Since $\mathbb{S}$ fulfils condition (SD), we conclude that $B = \emptyset$ and consequently, $\psi[a,b] \leq V_{[a,b]}(\psi, \mathbb{S}^+)$. □

We recall the definitions of lower and upper ($\mathbb{S}$)-limits, lower and upper ($\mathbb{S}$)-derivates and Theorem 54.5 from Thomson's book [3]. Set

$$\begin{aligned}
(\mathbb{S}) - \liminf_{y \to x} f(y) &= \sup\{t;\ \{x\} \cup f^{-1}(t, \infty) \in \mathbb{S}(x)\}, \\
(\mathbb{S}) - \limsup_{y \to x} f(y) &= \inf\{t;\ \{x\} \cup f^{-1}(-\infty, t) \in \mathbb{S}(x)\}, \\
(\mathbb{S}) - \underline{D}f(x) &= (\mathbb{S}) - \liminf_{y \to x} \frac{f(y) - f(x)}{y - x}, \\
(\mathbb{S}) - \overline{D}f(x) &= (\mathbb{S}) - \limsup_{y \to x} \frac{f(y) - f(x)}{y - x}
\end{aligned}$$

Theorem B *Let $\mathcal{T}$ be a collection of nonnegative subadditive interval functions, and let $\mathbb{S}$ be a local system such that $\psi(I) \leq V_I(\psi, \mathbb{S})$ for every interval I and any $\psi \in \mathcal{T}$. Let f be a real function with the following properties:*

(i) *$\Delta f^- \in \mathcal{T}$ (where $\Delta f^-[a,b] = \max\{f(a) - f(b), 0\}$),*

(ii) ($\mathbb{S}$) $-$ $\underline{D}f \geq 0$ *almost everywhere,*

(iii) $(\mathbb{S}) - \underline{D}f > -\infty$ ν_f*-almost everywhere (where* $\nu_f(E) = V(\Delta f^-, \mathbb{S}, E)$*).*

Then f *is nondecreasing.*

Theorem 2 *Let* $\mathbb{S}$ *be a local system satisfying condition (SD) and let* f *be a real function with the following properties:*

(a) $(\mathbb{S}^-) - \limsup_{y\to x} f(y) \le f(x)$ *for every* x*,*

(b) $(\mathbb{S}^+) - \underline{D}f \ge 0$ *almost everywhere,*

(c) $(\mathbb{S}^+) - \underline{D}f > -\infty$ *everywhere except possibly at points of a denumerable set, every point* x *of which satisfies the inequality* $f(x) \le (\mathbb{S}^+) - \liminf_{y\to x} f(y)$.

Then f *is nondecreasing.*

PROOF. Put $\psi[a,b] = \Delta f^-[a,b] = \max\{f(a) - f(b), 0\}$. Then ψ is a nonnegative subadditive interval function. We will show that the assumptions of Theorem B are satisfied (for the collection $\mathcal{T} = \{\psi\}$ and the local system $\mathbb{S}^+$). From (a) it follows that, for every x and each positive ε, there is $S \in \mathbb{S}^-(x)$ such that $f(y) < f(x) + \varepsilon$ for $y \in S$. Let $S_1 = S \cap (-\infty, x]$. Then $S_1 \in \mathbb{S}^-(x)$ and $\psi[y,x] < \varepsilon$ whenever $y \in S_1$. Thus ψ is $(\mathbb{S}^-)$-continuous and the lemma guarantees that $\psi(I) \le V_I(\psi, \mathbb{S}^+)$ for every interval I. This means that condition (i) of Theorem B holds. As (b) implies (ii), it suffices to show that (iii) results from (c).

Let $\varepsilon > 0$ and let x be a point of the set $E = \{x;\ (\mathbb{S}^+) - \underline{D}f(x) = -\infty\}$. (If $E = \emptyset$, then there is nothing to prove.) Since $f(x) \le (\mathbb{S}^+) - \liminf_{y\to x} f(y)$, there exists $S \in \mathbb{S}^+(x)$ such that $f(x) - \varepsilon < f(y)$ for $y \in S$. Put $S_1 = S \cap [x, \infty)$. Evidently, $S_1 \in \mathbb{S}^+(x)$. Moreover, because the family $C = \{[x,y];\ y \in S_1 \setminus \{x\}\}$ is an $(\mathbb{S}^+)$-cover of $\{x\}$, and each family of nonoverlapping intervals from C is one-element. Therefore

$$\nu_f(\{x\}) = V(\Delta f^-, \mathbb{S}^+, \{x\}) \le \mathrm{Var}\,(\psi, C) = \sup\{\psi[x,y];\ y \in S_1\} \le \varepsilon.$$

By the arbitrariness of ε, we get $\nu_f(\{x\}) = 0$. As E is denumerable, condition (iii) of Theorem B holds. This completes the proof because Theorem B implies that f is nondecreasing. □

Corollary 3 *Let* $\mathbb{S}$ *be a local system satisfying condition (SD) and let* f *be a real function with the following properties:*

(a) $(\mathbb{S}^-) - \limsup_{y\to x} f(y) \le f(x)$ *for every* x*,*

(b) $(\mathbb{S}^+) - \underline{D}f \ge 0$ *almost everywhere,*

(c) $(\mathbb{S}^+) - \underline{D}f > -\infty$ *everywhere.*

Then f is nondecreasing.

Corollary 4 *Let $\mathbb{S}$ be a local system satisfying condition (SD) and let f be a real function with the following properties:*

(a) $(\mathbb{S}^-) - \limsup_{y\to x} f(y) \le f(x) \le (\mathbb{S}^+) - \liminf_{y\to x} f(y)$ *for every x,*

(b) $(\mathbb{S}^+) - \underline{D}f \ge 0$ *almost everywhere,*

(c) $(\mathbb{S}^+) - \underline{D}f > -\infty$ *nearly everywhere.*

Then f is nondecreasing.

Corollary 5 *Let $\mathbb{S}$ be a local system satisfying condition (SD) and let f be a real function such that $(\mathbb{S}) - \underline{D}f(x) \ge 0$ for every x. Then f is nondecreasing.*

PROOF. Let $\varepsilon > 0$ let $g(x) = f(x) + \varepsilon x$. Then $(\mathbb{S}) - \underline{D}g(x) \ge \varepsilon > 0$ and hence there is $S \in \mathbb{S}(x)$ with $\dfrac{g(y) - g(x)}{y - x} > 0$ for $y \in S \setminus \{x\}$. Put $S_1 = S \cap (-\infty, x]$. Then $S_1 \in \mathbb{S}^-(x)$ and $g(x) > g(y)$ whenever $y \in S_1 \setminus \{x\}$. Thus $(\mathbb{S}^-) - \limsup_{y\to x} g(y) \le g(x)$ and since $(\mathbb{S}^+) - \underline{D}g \ge (S) - \underline{D}g > 0$, Theorem 2 implies that g is nondecreasing. Consequently, by the arbitrariness of ε, it follows that f is nondecreasing. □

Now we formulate a generalization of Theorem 4.3 from paper [2]. Our proof is almost identical with that in [2].

Theorem 6 *Let S be a local system satisfying condition (SD) and let f be a real function with the following properties:*

(a) $(\mathbb{S}^-) - \limsup_{y\to x} f(y) \le f(x) \le (\mathbb{S}^+) - \liminf_{y\to x} f(y)$ *for every x,*

(b) *$f(E)$ has void interior, where*

$$E = \{x;\ (\mathbb{S}^+) - \underline{D}f(x) \le 0 \ \textit{and}\ (\mathbb{S}^-) - \underline{D}f(x) \le 0\}.$$

Then f is nondecreasing.

PROOF. Suppose to the contrary that $f(a) > f(b)$ for some points a and b where $a < b$. Since $f(E)$ has empty interior, we can choose a point $k \notin f(E)$ for which $f(a) > k > f(b)$. Put

$$\begin{aligned} A &= \{x \in [a,b];\ f(x) > k \ \text{ or } \ f(x) = k \ \text{ and } \ (\mathbb{S}^+) - \underline{D}f(x) > 0\}, \\ B &= [a,b] \setminus A. \end{aligned}$$

First of all, observe that (b) implies $(\mathbb{S}^-) - \underline{D}f(x) > 0$ whenever $x \in B$ and $f(x) = k$. Let $x \in A$. If $f(x) > k$, then by (a) it follows that $(\mathbb{S}^+) - \liminf_{y \to x} f(y) \geq f(x) > k$. Hence there is $S \in \mathbb{S}^+(x)$ such that $f(y) > k$ for $y \in S$. Thus $S \subset A$ and therefore, $A \in \mathbb{S}^+(x)$. On the other hand, if $f(x) = k$, then $(\mathbb{S}^+) - \underline{D}f(x) > 0$. Thus there is $S \in \mathbb{S}^+(x)$ such that $S \subset [x, b]$ and $\dfrac{f(y) - f(x)}{y - x} > 0$ for $y \in S \setminus \{x\}$. So we have $f(y) > f(x) = k$. Hence $S \subset A$ and consequently, $A \in \mathbb{S}^+(x)$ in that case also.

Analogously we prove that $B \in \mathbb{S}^-(x)$ whenever $x \in B$. Since, evidently, $a \in A$ and $b \in B$, we arrive at a contradiction to condition (SD) which completes the proof. □

Thomson showed in his book [3] that many monotonicity theorems hold for local systems satisfying the intersection condition. Now we prove that this condition implies condition (SD).

We say that a local system $\mathbb{S}$ satisfies the intersection condition if, for any choice of sets $\{S_x;\ x \in \mathbb{R}\}$ with $S_x \in \mathbb{S}(x)$, there is a positive function δ on $\mathbb{R}$ such that $S_x \cap S_y \cap [x, y] \neq \emptyset$ whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$.

Theorem 7 *If a local system $\mathbb{S}$ is bilateral and satisfies the intersection condition, then $\mathbb{S}$ fulfils condition (SD).*

PROOF. Suppose that $\mathbb{S}$ is bilateral and satisfies the intersection condition but does not satisfy condition (SD). Then there are a closed interval $[a, b]$ and nonempty sets $A \subset [a, b]$, $B = A \setminus [a, b]$ for which conditions (a) - (c) (of Definition 1) hold. Without loss of generality we can assume that $b \in B$. For each $x \in [a, b]$, we put

$$S_x = \begin{cases} A \cup (-\infty, x) & \text{for } x \in A, \\ B \cup (x, \infty) & \text{for } x \in B. \end{cases}$$

Obviously, $S_x \in \mathbb{S}(x)$ for any $x \in [a, b]$. Thus we can find a positive function δ such that $S_x \cap S_y \cap [x, y] \neq \emptyset$ whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$. Let $a_1 \in A$ be a right-hand accumulation point of B. Such a point exists. If a does not fulfil this condition, we can choose as a_1 the right endpoint of the component of A which contains a. In a similar way we find $b_1 \in B \cap (a_1, a_1 + \delta(a_1))$ which is a left-hand accumulation point of A. Suppose we have already chosen an increasing sequence $\{a_1, a_2, \ldots, a_n\} \subset A$ and a decreasing sequence $\{b_1, \ldots, b_n\} \subset B$ with $a_n < b_n$. Then for a_{n+1} we take a point from A which is a right-hand accumulation point of B and satisfies the inequalities $a_n < a_{n+1} < b_n$ and $b_n - a_{n+1} < \min\{\delta(b_n), \frac{1}{n}\}$. Similarly we choose $b_{n+1} \in B$ such that b_{n+1} is a left-hand accumulation point of A, $a_{n+1} < b_{n+1} < b_n$ and $b_{n+1} - a_{n+1} < \min\{\delta(a_{n+1}), \frac{1}{n+1}\}$.

Let $c = \lim a_n = \lim b_n$. Then, for every positive integer n, we have

$$0 < c - a_n < b_n - a_n < \delta(a_n),$$
$$0 < b_n - c < b_n - a_{n+1} < \delta(b_n).$$

Since $c \in (a, b)$, c belongs either to A or to B. Assume that $c \in A$. (In case $c \in B$, the proof is similar.) Thus there is a positive integer n with $b_n \in (c, c + \delta(c))$. Hence $0 < b_n - c < \min\{\delta(b_n), \delta(c)\}$ and, therefore, $S_c \cap S_{b_n} \cap [c, b_n] \neq \emptyset$. So, we have

$$\begin{aligned} \emptyset &\neq [A \cup (-\infty, c)] \cap [B \cup (b_n, \infty)] \cap [c, b_n] \\ &= A \cap B \cap [c, b_n] \subset A \cap B = \emptyset. \end{aligned}$$

This contradiction completes the proof. □

It is easy to show that converse to Theorem 7 is not true. This results from the following example.

Example Let

$$\mathbb{S}(x) = \{A \subset \mathbb{R};\ x \in A,\ \underline{d}^i_+(A, x) > 0 \ \text{ and } \ \underline{d}^i_-(A, x) > 0\}.$$

Theorem A implies that $\mathbb{S}$ fulfils condition (SD). But from [1, Theorem 2] it follows that $\mathbb{S}$ does not satisfy the intersection condition.

References

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