RESEARCH ARTICLES

Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 59–80

Henry Fast, Department of Mathematics, Wayne State University, Detroit, MI 48202

INVERSION OF THE CROFTON TRANSFORM FOR SETS IN THE PLANE

Abstract

The genesis of this paper goes back to a question posed years ago by the late H. Steinhaus, "About two plane arcs both of finite lengths it is known that every line in the plane meets both at the same number of points (which may be zero or ∞). Must the arcs be identical ?" This question in answered in this paper using the Crofton transform. For a fixed set in the plane the Crofton transform for that set is defined as the number of points in which the set meets a variable line in the plane. In this paper we construct an inversion of the Crofton transform within a certain class of plane Borel sets, a somewhat weak form of inversion retrieving from the Crofton transform (actually from the cross-integral function, which is somewhat stronger) the closure of the set rather than the set (or the set modulo a set of linear measure zero) itself. We also establish uniqueness (or rather a degree of uniqueness) of two plane sets from the same class, whose Crofton transforms coincide over certain families of lines. This answers a stronger version of the original Steinhaus question.

1. Preliminaries

The setting of our considerations is $\mathbb{R}^2$ which for convenience is assumed to be the complex plane. Throughout the paper we adhere to the following customary notation and terminology for sets A, $E \subset \mathbb{R}^2$ and $z \in \mathbb{R}^2$.

$E + z$ denotes the z translate of E.

$\delta(z, E)$ denotes the distance from z to a set E.

$\delta(A, E)$ denotes the distance between A and E.

Mathematical Reviews subject classification: 28A75

Received by the editors February 4, 1992

diam (E) denotes the diameter of E.

$U(z, r)$ denotes $\{w \in \mathbb{R}^2 : |w - z| < r\}$.

C denotes the unit circle, $C = \partial U(0,1) = \{\nu \in R^2 : |\nu| = 1\}$.

dom Φ denotes the domain of the function Φ.

We shall casually use the same notation $|E|_k$ for k dimensional, $k = 0, 1, 2, \ldots$ (Hausdorff, in particular Lebesgue) measure of a set E regardless of the particular space in which the measure is considered. (For $k = 0$ the measure is the so-called *counting measure.*)

Also it is necessary to adopt some additional notation, which is listed here.

Bor (X) denotes the class of Borel subset of a (topological) space X.

If $D \subset A \times B$ and if $a \in A$, $b \in B$, then the *a-section* and *b-section* of D are the sets $D_a = \{y : (a, y) \in D\}$ and $D^b = \{x : (x, b) \in D\}$.

L denotes the set of all the straight lines in $\mathbb{R}^2$.

ℓ denotes an individual line from L.

$\angle[\ell_1, \ell_2]$ denotes the smaller of the angles formed by the two lines $\ell_1, \ell_2 \in L$. Note that $0 \leq \angle[\ell_1, \ell_2] < \pi/2$.

L^0 denotes the subset of L consisting of lines passing through 0.

ℓ^0 denotes an individual line from L^0. In particular the same notation is used for the line from L^0 which is parallel to a given $\ell \in L$ (the projection of ℓ into L^0).

$\ell(E, z)$ denotes the line tangent to a set $E \subset \mathbb{R}^2$ at a point $z \in E$ if the tangent line exists at z.

$\ell^\perp$ denotes the line from L^0 which is perpendicular to $\ell \in L$.

$w(\ell) \in \mathbb{R}^2$ denotes the point of intersection of $(\ell^0)^\perp$ and ℓ.

For $z \in \mathbb{R}^2$ and $\nu \in C$ we write $\ell(z, \nu)$ for the line through z in the direction ν. Note that $\ell(z, \nu) = \ell(z, -\nu)$ and $L^0 = \{\ell(0, \nu) : \nu \in C\}$. The set L^0 can be metrized by letting $\angle[\ell, \ell']$ be the distance between the lines ℓ and ℓ'. The product $L^0 \times \mathbb{R}^2$ (which is locally $\mathbb{R}^3$) carries the measures of $\mathbb{R}^3$, $|\cdot|_k$, $k = 0, 1, 2, 3$. We have a one-to-one mapping $\ell \to (\ell^0, w(\ell))$ of L into $L^0 \times \mathbb{R}^2$. The image of L in this product is a two dimensional surface (actually, a Möbius strip) M. Let $\bar{\ell} \in L^0$. It is easy to see that the fiber $\{\ell \in L : \ell^0 = \bar{\ell}\}$

of M is the straight line $\bar{\ell} \times \bar{\ell}^{\perp} \subset L^0 \times \mathbb{R}^2$. The one-to-one mapping between L and M can now be used to induce a metric and measures, $|\cdot|_k$; $k = 0, 1, 2$, on L from M. Note that convergence $\lim_{n\to\infty} \ell_n = \ell$ in the induced metric is convergence in the set theoretical sense. A sequence, $\{A_n\}$, of sets converges in the set theoretic sense to the set A means if $a_n \in A_n$ for each $n \in \mathbb{N}$ and if $\lim_{n\to\infty} a_n = a$, then $a \in A$ and conversely every $a \in A$ is the limit of such a sequence.

Next we recall the following result due to Besicovitch concerning the structure of a linearly measurable subset of the plane of finite linear measure. (See [2] and [3].)

Such a set can be decomposed into two disjoint parts, a regular part and an irregular part. The regular part is a subset of a union of at most a countable family of arcs each two of which are disjoint except possibly for their endpoints and the sum of whose lengths is finite. The irregular part projects in almost every direction onto a set of linear measure zero and also intersects with any arc of finite length in a set of linear measure zero.

Note that since a subset of linear measure zero can be freely moved from the regular to the irregular part, this decomposition is unique only up to sets of linear measure zero.

A set identical with its regular part shall be referred to as *B-regular*; when identical with its irregular part, as *B-irregular*. For regular and irregular parts of E we use the notation:$\mathfrak{R}(E)$ and $\mathfrak{I}(E)$.

We apply the following special terminology for a set $E \subset \mathbb{R}^2$.

* E is *linear* if it is a subset of a line called its *carrier*, denoted by $\ell(E)$.

* E is *multilinear* if it is a union of finitely many disjoint linear sets, its *subsets components.*

* E is *metrically dense* (in linear measure) at a point z if $|U(z,r) \cap E|_1 > 0$ for every $r > 0$. (Also see [1].)

* E is *metrically dense in itself* if it is metrically dense at each of its points. (Thus $\emptyset$ is metrically dense in itself.) (In the linear case, see [1].)

* E is *rectifiable* if it is linearly measurable with $|E|_1 < \infty$, and B-regular. It is *strongly rectifiable* if in addition $|\mathrm{cl}\,(E)|_2 = 0$.

(Note that a strongly rectifiable set is nowhere dense in $\mathbb{R}^2$.)

The result of this paper applies to the class of those Borel subsets of finite linear measure in the plane whose B-regular parts are strongly rectifiable. (However, in what follows no assumptions are made beyond what is actually needed to prove each particular assertion.)

For a rectifiable set, E, recall that for almost every (in linear measure) point $z \in E$ the tangent $\ell(E, z)$ exists. (Again see [2] and [3].) Accordingly almost every point of E falls into exactly one of the following three sets:

* $SL(E) = \{z \in E : |U(z,r) \cap \ell(E,z) \setminus E|_1 = 0 \text{ for some } r > 0\}$. Such points are called *strongly linear.*

* $NL(E) = \{z \in E : |U(z,r) \cap \ell(E,z) \setminus E|_1 = 2r \text{ for some } r > 0\}$. Such points are called *non-linear.*

* $WL(E) = \{z \in E : 0 < |U(z,r) \cap \ell(E,z) \setminus E|_1 < 2r \text{ for all } r > 0\}$. Such points are called *weakly linear.*

In addition

* $z \in E$ is said to be *metrically laterally non-isolated* if the set $E \setminus \ell(E, z)$ is metrically dense at z; otherwise, it is *metrically laterally isolated.*

Write SLN (E) and SLI (E) for the two complementary subsets of $SL(E)$ made of metrically laterally non-isolated and metrically laterally isolated points.

Definition 1 *Let $E \subset \mathbb{R}^2$ be a measurable rectifiable set and let $\delta(z, E) > 0$. The tangential support set, $Ts(E, z)$, of E at z is defined by*

$$Ts(E, z) = \{u \in E : z \in \ell(E, u)\}.$$

It is not hard to see that

$$\mathrm{Ts}\,(E, z) = \{u \in E : z - u \text{ is parallel to } \ell(E, u)\}.$$

Definition 2 *Let $E \subset \mathbb{R}^2$ be measurable. Then the spectrum of E, $Sp(E)$, is defined by $Sp(E) = \{z \in \mathbb{R}^2 : |ts(E, z)|_1 > 0\}$.*

Note: Let E be a rectifiable set. Then $SL(E) \cup WL(E) \subset E \cap \mathrm{Sp}(E)$ and $\cup\{\ell : |\ell \cap E|_1 > 0\} \subset Sp(E)$. Also $\mathrm{Sp}(E)$ contains at most countable many lines.

Definition 3 *A regular multidomain is a finite union of bounded closed domains whose boundaries consist of piecewise smooth arcs not containing a straight line segment.*

2. A Few Preparatory Lemmas

Lemma 2.1 *A linearly measurable set E is a union of a set metrically dense in itself and a set of (linear) measure zero.* (Compare to [1], page 138.)

PROOF. If E is not metrically dense at z, then there is a $r > 0$ such that $E \cap U(z,r)$ is of linear measure zero. Select a rational $z' \in \mathbb{R}^2$ and a rational $r' > 0$ for which $z \in U(z',r') \subset U(z,r)$. Then $E \cap U(z',r')$ is of linear measure zero. The (countable) union of all such portions, $E \cap U(z',r')$, is of linear measure zero. After removal of this part from E what remains is a subset of E which is metrically dense at each of its points. □

Observe that to an E which has a closed B-regular part one may uniquely assign a B-regular part which is metrically dense in itself. By Lemma 2.1 it suffices to delete from that closed part the non dense in itself subset of linear measure zero (moving it to the B-irregular part) and subsequently close again what remains.

In the sequel we shall make use of the following lemma dealing with a property of measurable subsets of the real line

Lemma 2.2 *Let $E \subset \mathbb{R}$ be measurable and let $J = (a,b)$. Suppose both E and its complement are of positive measure in J; that is, $0 < |J \cap E|_1 < |J|_1$. Then* $\liminf_{t\to 0} \dfrac{|J \cap ((E+t)\setminus E)|_1}{t} \geq 1$.

PROOF. Let x_0, $y_0 \in (a,b)$ have metric density one and zero respectively in E. Without loss of generality assume that $x_0 < y_0$. Let $\epsilon > 0$. There is $h \in (0, x_0 - a)$ so small that $|(x_0 - t/2, x_0 + t/2) \cap E| > (1-\epsilon)t$ and $|(y_0 - t/2, y_0 + t/2) \cap E| < \epsilon t$ for $0 < t < h$. Select $n \in \mathbb{N}$ so large that $\frac{(y_0 - x_0)}{n} < h$. Let $t = \frac{(y_0-x_0)}{n}$ and set $q_k = x_0 - \frac{t}{2} + kt$ for $k = 0, 1, \ldots, n+1$ thus partitioning the interval $J^* = [x_0 - t/2, y_0 + t/2)$ into $n+1$ subintervals of equal length t. The interval $[q_0, q_1) = [x_0 - t/2, x_0 + t/2)$ is the first and $[q_n, q_{n+1}) = [y_0 - t/2, y_0 + t/2)$ is the last of them. Add an extra interval $[q_{-1}, q_0)$ at the beginning by setting $q_{-1} = q_0 - t$. Accordingly $J^* \cap E$ is partitioned into the $n+1$ subportions $E_k = [q_k, q_{k+1}) \cap E$; $k = 0, 1, \ldots, n$. Add the extra subportion, $E_{-1} = [q_{-1}, q_0) \cap E$. We have $J^* \cap E = \cup_{k=0}^{n} E_k$ and $J^* \cap (E+t) = \cup_{k=-1}^{n-1}(E_k + t) = \cup_{k=0}^{n}(E_{k-1}+t)$ and for their difference $J^* \cap ((E+t)\setminus E) = \cup_{k=0}^{n}\left((E_{k-1}+t)\setminus E_k\right)$. For each $k = 1, \ldots, n$ shift $[q_k, q_{k+1})$, so that it will overlap with the last interval $[y_0 - t/2, y_0 + t/2)$. The portions $(E_{k-1}+h)\setminus E_k$ shifted along with the intervals containing them become subsets of the last interval and actually, jointly with the last portion E_n (which remained in place) they form a partition of the (shifted) first portion E_0 into disjoint sets.

This fact yields

$$|E_0|_1 = \sum_{k=1}^{n} |(E_{k-1}+t) \setminus E_k|_1 + |E_n|_1.$$

Thus we obtain

$$\begin{aligned} |E_0|_1 - |E_n|_1 &= |\cup_{k=1}^{n}(E_{k-1}+t) \setminus E_k|_1 \\ &\leq |J^* \cap ((E+t) \setminus E)|_1 \leq |J \cap ((E+h) \setminus E)|_1 \end{aligned}$$

and since $|E_0|_1 - |E_n|_1 \geq (1-\epsilon)t - \epsilon t$ and $|E_{-1}|_1 \leq t$,

$$\frac{|J \cap ((E+t) \setminus E))|_1}{t} \geq 1 - 2\epsilon$$

which in view of arbitrariness of ϵ proves the lemma. □

Definition 4 *A function f with with $\operatorname{dom} f$ open in $\mathbb{R}^2$ is d-smooth if at every $z \in \operatorname{dom} f$ it has directional derivative in every direction. A function $\hat{f}$ of the form $\hat{f} = f + f'$ where f is d-smooth and $f' = 0\,|\cdot|_2$ almost everywhere in $\mathbb{R}^2$ is d-smooth equivalent.*

For the final lemma of this section we introduce three differential operators acting on d-smooth equivalent functions.

Definition 5 *Let $\hat{f}$ be a function with $\operatorname{dom}\hat{f}$ open in $\mathbb{R}^2$, let $z \in \mathbb{R}^2$ and let $\nu \in C$.*

(i) $D_\nu \hat{f} = \lim_{t\to 0+} \dfrac{\hat{f}(z+t\nu) - \hat{f}(z)}{t}$.

(ii) *For $\ell = \ell(z,\nu)$ let $S_\ell \hat{f}(z) = D_\nu \hat{f}(z) + D_{-\nu}\hat{f}(z)$.*

(iii) *For $\ell = \ell(z,\nu)$ let*

$$\overline{S}_\ell \hat{f}(z) = \limsup_{t\to 0+} \frac{f(z+t\nu) + f(z-t\nu) - f(z+0\nu) - f(z-0\nu)}{t},$$

where $f(z \pm 0\nu) = \lim_{t\to 0+} f(z \pm t\nu)$.

Note that if f is differentiable at z, then $S_\ell \hat{f}(z) = 0$ for all $\ell \in L$. Also $\overline{S}_\ell$ does not require $\hat{f}$ to be defined at z. It suffices that z is a limit point of $\operatorname{dom} f$ which contains $U(z,r) \cap \ell \setminus \{z\}$.

Clearly for a d-smooth function $D_\nu)$ is the one sided directional derivative operator and $\overline{S}_\ell = S_\ell$.

The *kernel function* ϕ introduced below will play the role of an integral kernel in the sequel. It is defined in terms of the function ψ introduced next.

Definition 6 *For $z \in \mathbb{R}^2 \setminus \{0\}$ and for $\mu \in C$ set $\psi(z,\mu) = Im\left(\frac{\mu}{z}\right)$.*

Note that ψ is harmonic in z with a pole at 0 and vanishes along its *null line*, $\ell(0,\mu) = \{z : z/|z| = \pm\mu, z \neq 0\} = \ell^0(0,\mu)$. Moreover $\psi(z,\mu) = -\sin(\arg z - \arg\mu)/|z|$ and $\psi(z,\mu) = -\psi(z,-\mu)$.

Definition 7 *Let $z \in \mathbb{R}^2$ and let $\ell \in L$ with direction $\mu \in C$. Set*

$$\phi(z,\ell) = |\psi(z,\mu)| = \frac{|\sin(\arg z - \arg\mu)|}{|z|} = \frac{\sin\angle[\ell(0,z/|z|),\ell]}{|z|}.$$

Clearly $\phi(z,\ell) \geq 0$, has the same null line as $\psi(z,\mu)$, which for ϕ is its line of symmetry, is continuous in its domain, is harmonic in z outside its null line and $\phi(z,\ell) \leq |z|^{-1}$. The following lemma contains a few more properties of both functions, ψ and ϕ.

Lemma 2.3 *Let $u, z \in \mathbb{R}^2$, let ν, $\mu \in C$ and let $\ell \in L$.*

(a) *For $u \in \ell$ $\phi(u-z,\ell)$ is harmonic in z for $z \notin \ell$.*

(b) *Let $u \neq z$ and let $\ell = \ell(u,\nu)$. Then $D_\nu \arg(u-z) = \psi(u-z,\nu)$ and $|D_\nu \arg(u-z)| = \phi(u-z,\ell)$.*

(c) *Let $u \neq z \in \ell \in L$. Then* $\nabla_u \psi(u-z,\mu) = -\dfrac{i\exp(2i(\arg(u-z)+\arg\mu))}{|u-z|^2}.$

(d) *Let u be on the null line with $u \neq z$. Then*

$$\lim_{u'\to u, \angle[\ell',\ell]\to 0} D_\nu\phi(u'-z,\ell') = D_\nu\phi(u-z,\ell) \qquad (2.3.d*)$$

and

$$|D_\nu\phi(u-z,\ell)| \leq \frac{1}{|u-z|^2}. \qquad (2.3.d**)$$

(e) *Let u be on the null line with $u \neq z$. Then*

$$D_\nu\phi(u-z,\ell) = D_{-\nu}\phi(u-z,\ell) = \frac{|\sin(\arg(u-z)-\arg\nu)|}{|z-u|^2.}.$$

(f) *Let u be on the null line with $u \neq z$. Then for $\ell' \in L^0$ $S_{\ell'}\phi(u-z,\ell) = \dfrac{2\cos\angle[\ell,\ell']}{|u-z|^2}$. (In particular $S_{\ell^\perp}\phi(u-z,\ell) = \dfrac{2}{|u-z|^2}$.)*

PROOF. (a) Replacing z with $u-z$ one observes that for $z \notin \ell$ $\phi(u-z, \ell(z,\mu)) = \pm\psi(u-z,\mu)$ where the sign is $+$ or $-$ depending on which side of the null line the point u lies. (Note: On the null line ϕ is continuous but not differentiable.)

(b) It is elementary to establish that $\nabla_u(\arg(u-z)) = i(u-z)|u-z|^{-2}$. Consequently,

$$D_\nu \arg(u-z) = |\nabla_u \arg(u-z)| \cdot \cos\angle(\nabla_u(\arg(u-z)), \nu) =$$

$$\frac{\sin(\arg(u-z) - \arg\nu)}{|u-z|} = \psi(u-z,\nu).$$

Hence $|D_\nu \arg(u-z)| = |\psi(u-z,\nu)| = \phi(u-z,\ell)$ where $\ell = \ell(0,\nu)$.

(c) By elementary calculation

$$\frac{\partial}{\partial x}\frac{\Im(z)}{|z|^2} = -\frac{2\Im(z)\cdot\Re(z)}{|z|^4} = \frac{-2\sin\arg(z)\cdot\cos\arg(z)}{|z|^2} = \frac{-\sin(2\arg(z))}{|z|^2}$$

and

$$\frac{\partial}{\partial y}\frac{\Im(z)}{|z|^2} = \frac{(\Re(z))^2 - (\Im(z))^2}{|z|^4} = \frac{\cos^2\arg(z) - \sin^2\arg(z)}{|z|^2} = \frac{\cos(2\arg(z))}{|z|^2}.$$

Thus

$$\nabla\frac{\Im(z)}{|z|^2} = \frac{-\sin(2\arg(z)) + i\cos(2\arg(z))}{|z|^2} = \frac{i\exp(2i\arg z)}{|z|^2}$$

and $\psi(z,1) = Im(1/z) = -Im(z)/|z|^2$ yields $\nabla\psi(z,1) = -i\exp(2i\arg z)/|z|^2$. Clearly, for $u, z \in \mathbb{R}^2$, $u \neq z$ we obtain the expression as stated in (c).

(d) (2.3.d*) is easily verified from the definition of ϕ and of the operators involved. The estimate (2.3.d**) follows directly from part (c) of this lemma.

(e) $\psi(u-z,\mu)$ is differentiable on its null line for $u \neq z$ and also antisymmetric with respect to it. Thus for u situated on that line $D_\nu\psi(u-z,\mu) = -D_\nu\psi(u-z,\mu)$ which implies the first equality of (e) for $\phi = |\psi|$. From part (c) of this lemma we get that for the points u on the null line; that is, points satisfying $\arg(u-z) = \pm\mu$, $\nabla\psi$ is perpendicular to the null line from which (e) is a direct consequence.

(f) This follows immediately from part (e) of this lemma. □

3. The Crofton Transform and the Cross Integral. More about L and More Preparatory Lemmas.

Definition 8 *Let $E \subset \mathbb{R}^2$. The Crofton transform (Crofton function) of E, N_E, is defined by $N_E(\ell) = |\ell \cap E|_o$. (See [4].)*

Thus N_E maps L into $\{0\} \cup \mathbb{N} \cup \{0, \infty\}$ counting the number of points in $\ell \cap E$. Note that $N_E(\ell) = \int_\ell \chi_E(u)\,|du|_0$.

We need certain known properties of the Crofton functions of Borel sets. Their proof are included to make the paper as self-contained as possible. The next three lemmas deal with sets of lines in L and with the properties of the Crofton function. The first uses the continuous mapping $T : L^0 \times \mathbb{R}^2 \to L$ defined by $T(\ell, z) = z + \ell$ and exploits the one-to-one mapping of L into $L^0 \times \mathbb{R}^2$.

Lemma 3.1 (a) *Let $E \in \mathrm{Bor}\,(\mathbb{R}^2)$. The support, $\mathrm{supp}\,N_E$, of the Crofton function of E is $|\cdot|_2$ measurable in L. The sections $(\mathrm{supp}\,N_E)_{\bar\ell} = \{\ell \in \mathrm{supp}\,N_E : \ell^0 = \bar\ell\}$ and $(\mathrm{supp}\,N_E)^z = \{z + \ell^0 \in \mathrm{supp}\,N_E : \ell^0 \in L^0\}$ are $|\cdot|_1$ measurable in their respective spaces.*
(b) *For $A \in \mathrm{Bor}\,(L)$ the section $(T^{-1}(A))^z = \{\ell^0 \in L^0 : z + \ell^0 \in A\}$ is in $\mathrm{Bor}\,(L^0)$ for every z. Moreover, $|A|_2 = 0$ if and only if $|(T^{-1}(A))^z|_1 = 0$ for $|\cdot|_2$ almost every $z \in \mathbb{R}^2$.*

PROOF. (a) For $z \in E$ and $\ell^0 \in L^0$ evidently $z + \ell^0 \in \mathrm{supp}\,N_E$. On the other hand if $\ell \in \mathrm{supp}\,N_E$, then there is a $z \in E$ such that $z + \ell^0 \in \mathrm{supp}\,N_E$. This means that $\mathrm{supp}\,N_E$ is the T-image of the set $L^0 \times E$. A continuous image of a Borel set is a Suslin (analytic) set. So $\mathrm{supp}\,N_E$ is a Suslin set in L. Its $\bar\ell$ sections and z sections are intersections of $\mathrm{supp}\,N_E$ with closed subsets of L and hence Suslin sets. It is known that Suslin sets are measurable. Thus $\mathrm{supp}\,N_E$ and both types of sections are respectively two and one dimensionally measurable in their respective spaces.

(b) Set $B = T^{-1}(A)$. By Fubini's Theorem $|A|_2 = 0$ if and only if $|A_{\ell^0}|_1 = 0$ for $|\cdot|_1$ almost every ℓ^0. Observe that $B_{\ell^0} = \{(z, \ell^0) : z + \ell^0 \in A\} = \cup\{(z, \ell^0) : z \in \ell \in A_{\ell^0}\}$. Thus $|B_{\ell^0}|_2 = 0$ if and only if $|A_{\ell^0}|_1 = 0$. By Fubini's Theorem $|B|_3 = 0$ if and only if $|B_{\ell^0}|_2 = 0$ for $|\cdot|_1$ almost every ℓ^0. By the previous observations $|B|_3 = 0$ if and only if $|A|_2 = 0$. Also By Fubini's Theorem $|B|_3 = 0$ if and only $|B^z|_2 = 0$ for $|\cdot|_2$ almost every $z \in \mathbb{R}^2$. □

Lemma 3.2 *Let $E \in \mathrm{Bor}\,(\mathbb{R}^2)$. Then N_E is a Borel function on L. Moreover if E is rectifiable, then $\int_L N_E(\ell)\,|d\ell|_2 = 2|\mathfrak{R}(E)|_1$.* (This is known as the *Crofton formula.*)

PROOF. First show that if $E \in \mathrm{Bor}\,(\mathbb{R}^2)$, then N_E is a Borel function. For $m \in \mathbb{N}$ let $\{E_{k,m} : k = 1, 2 \ldots\}$ be a Borel partition of E into subsets with $\mathrm{diam}\,E_{k,m} \le 1/m$. Let $\chi_{k,m} : L \to \mathbb{R}$ be the characteristic function of $\mathrm{supp}\,N_{E_{k,m}}$. It is easy to see that the values of $\sum_k \chi_{k,m}$ do not decrease with increasing m. Actually when $N_E(\ell) < \infty$, they eventually equal $N_E(\ell)$; otherwise they tend to ∞. In other words, the nondecreasing sequence $\sum_k \chi_{k,m}$

of functions converges pointwise everywhere on L to N_E. By Lemma 3.1 (a) $\chi_{k,m}$ is a Borel function on L and for each $m \in \mathbb{N}$ so is $\sum_k \chi_{k,m}$. Thus N_E is a Borel function. This concludes the proof of the first part.

Let E be a subset of a rectifiable arc S and as before let $\{S_{k,m} : k = 1, 2, ...\}$ be a partition of S into subarcs with diam $(S_{k,m}) \le 1/m$. Let $J_{k,m}$ be the chord joining the endpoint of $S_{k,m}$ containing the first endpoint but not the second (except for the last which contains both endpoints). The union of these cords is a polygon S_m, inscribed in S. a straight line segment J The projection of the chord $J_{k,m}$ onto $\ell^0 \in L^0$ is a segment of length $|J_{k,m}|_1 \cdot |\cos\theta_{k,m}(\ell^0)|$ where $\theta_{k,m}(\ell^0)$ is the angle between $J_{k,m}$ and ℓ^0. Thus for each $\ell^0 \in L^0$ except for that ℓ^0 parallel to $J_{k,m}$, we have $\int_{\ell^0} N_{J_{k,m}}(u)\,|du|_1 = |J_{k,m}|_1 |\cos\theta_{k,m}(\ell^0)|$. Integrating with respect to ℓ^0 over L^0 yields $\int_L N_{J_{k,m}}(\ell)\,|d\ell|_2 = 2|J_{k,m}|_1$. By the choice of the cords $J_{k,m}$, we have $N_{S_m} = \sum_k N_{J_{k,m}}$ and, except for the countable set of lines containing the chords $J_{k,m}$, $N_{S_m}(\ell)$ does not decrease with increasing m. Moreover except for those lines that meet S locally in exactly one point, $N_{S_m}(\ell)$ tends to $N_S(\ell)$. These exceptional lines are known as support lines. The number of them in any one fixed direction is easily seen to be countable. Hence by Fubini's Theorem the set of support lines is of $|\cdot|_2$ measure 0. Thus N_{S_m} converges almost everywhere on L to N_S. So by the Lebesgue Monotone convergence theorem,

$$\int_L N_S(\ell)\,|d\ell|_2 = \lim_{m\to\infty} \int_L N_{S_m(\ell)}\,|d\ell|_2 = \lim_{m\to\infty} \sum_k 2|J_{k,m}|_1 = 2|S|_1$$

which verifies the Crofton formula thus far for an arc. From an arc, S, the formula extends to a union of disjoint subarcs of S and thus to open (relative to S) subsets of S. From there it extends to G_δ subsets of S using the Lebesgue Dominated Convergence Theorem. But this implies that the formula holds for every $E \in \mathrm{Bor}\,(S)$. Finally, the formula extends to Borel subsets of countable unions of rectifiable arcs each two having at most endpoints in common. Thus by the Besicovitch structure theorem, it extends to all rectifiable B-regular Borel subsets of $\mathbb{R}^2$.

The B-irregular part of E is also a Borel set. Hence its Crofton function is a Borel function. By the property characterizing the B-irregular part, its Crofton function vanishes $|\cdot|_1$ almost everywhere on every section $\{\ell : \ell^0 = const\}$ through L which implies $\int_{\{\ell:\ell^0=const\}} N_{\mathfrak{I}(E)}(w)\,|dw|_1 = 0$. By Fubini's Theorem $\int N_{\mathfrak{I}(E)}(\ell)\,|d\ell|_2 = 0$. □

Lemma 3.2 immediately implies the following corollary:

Corollary 3.3 *The Crofton function of a rectifiable Borel set is finite $|\cdot|_2$ almost everywhere on L.*

The function defined next is crucial for our considerations. It is called the *cross integral.*

Definition 9 *For a rectifiable $E \in \mathrm{Bor}(\mathbb{R}^2)$ and $z \in \mathbb{R}^2$ let*

$$C_E(z) = \int_{L^0} N_E(z + \ell^0)\, |d\ell^0|_1.$$

Lemma 3.4 (a) *Let $E \in \mathrm{Bor}(\mathbb{R}^2)$ be rectifiable. Then C_E is a Borel function on R^2. and $C_E(z) = C_{\mathfrak{R}}(z)$.*
(b) *Let E_1, $E_2 \in \mathrm{Bor}(\mathbb{R}^2)$ be rectifiable. Then the following are equivalent:*

(*) $N_{E_1}(\ell) = N_{E_2}(\ell)$ *for* $|\cdot|_2$ *almost every* ℓ

(**) $C_{E_1}(z) = C_{E_2}(z)$ *for* $|\cdot|_2$ *almost every* z.

PROOF. (a) The first part follows from general facts about product integration. Since $C_E = C_{\mathfrak{R}(E)} + C_{\mathfrak{I}(E)}$, it is enough to note that as a consequence of Lemma 3.2 $\int_{L^0} N_{\mathfrak{I}(E)}(z + \ell^0)\, |d\ell^0|_1 = 0$ for $|\cdot|_2$ almost every $z \in \mathbb{R}^2$ which proves (a).

(b) By (*) the set $A = \{\ell \in L : N_{E_1}(\ell) \neq N_{E_2}(\ell)\}$ is of $|\cdot|_2$ zero. Therefore by Lemma 3.1 (b) for $|\cdot|_2$ almost every $z \in \mathbb{R}^2$ the section $\{\ell^0 \in L^0 : z + \ell^0 \in A\}$ is of $|\cdot|_1$ zero. Thus both cross integral functions in (**) are equal for such z.

Conversely if (**) holds, then $|\cdot|_2$ almost every z−section through A mentioned above is of $|\cdot|_1$ zero. Thus $|A|_2 = 0$. □

4. The Full Angle of Visibility, Φ_E.

For $E \in \mathrm{Bor}(\mathbb{R}^2)$ rectifiable and $z \in \mathbb{R}^2$ we define $\Phi_E(z)$ which we call the *full angle of visibility of E from the point z.* (The intuition behind the choice of such a term will be clear after the proof of Lemma 4.2 (a).) It plays a crucial role in this paper.

Definition 10 *Let $E \in \mathrm{Bor}(\mathbb{R}^2)$ be rectifiable and let $z \in \mathbb{R}^2$. Then*

$$\Phi_E(z) = \int_E \phi(u - z, \ell(E, u))\, |du|_1.$$

The following lemma establishes certain properties of this function.

Lemma 4.1 *Let $E \in \mathrm{Bor}(\mathbb{R}^2)$ be rectifiable.* (a) *Φ_E is finite and continuous on $\mathbb{R}^2 \setminus cl(E)$. Moreover $\Phi_E(z) \le \dfrac{1}{\delta(z, E)}|E|_1$.*
(b) *$\Phi_E(z) = C_E(z)$ for $z \in \mathbb{R}^2 \setminus cl(E)$.*
(c) *If E is linear with carrier K, then Φ_E is harmonic on $\mathbb{R}^2 \setminus K$.*

Note: A rectifiable B-regular set E can be dense in $\mathbb{R}^2$ making the domain of continuity of Φ_E empty. For a strongly rectifiable E the $\mathbb{R}^2 \setminus E$ is an open dense subset of $\mathbb{R}^2$.

PROOF. First assume that E is a rectifiable arc, S, with arclength parametrization $u = u(s) : s \in [0, |S|_1]$. The assertion (b) will be established first in this special case. Let $z \in \mathbb{R}^2$ with $\delta(z, S) > 0$. The function $u(s)$ as well as $\arg(u(s) - z)$ are absolutely continuous on $[0, |S|_1]$. Consequently the variation of $\arg(u(s) - z)$ over that interval is $\int_0^{|S|_1} \left|\frac{d}{ds}\arg(u(s) - z)\right| ds$. Note that u' is the unit derivative vector of u. Using the notation introduced earlier we write $|(d/ds)\arg(u(s) - z)| = |D_{u'}\arg(u - z)|$. By Lemma 2.3 (b), $|D_{u'}.\arg(u - z)| = \phi(u - z, \ell(E, u))$. Thus the above integral equals $\int_E \phi(u - z, \ell(E, u))\, |du|_1$, which by definition is $\Phi_E(z)$. On the other hand the variation of $\arg(u(s) - z)$ can be represented in terms of the so-called *Banach indicatrix*, $\mathrm{Ind}\,(\theta) = |\{s \in S : \arg u(s) = \theta\}|_o$. (Here it is assumed that $\arg(u) \in [0, 2\pi]$.) In particular $\Phi_S(z) = \int_0^{2\pi} \mathrm{Ind}\,(\theta)\, d\theta$. It is easy to see that the right hand side of the expression above is the same as the cross integral, $C_S(z)$, for the arc S which was introduced earlier. Thus (b) holds for rectifiable arcs.

For (a) again in the special case of a rectifiable arc note that finiteness and continuity of Φ_S in $\mathbb{R}^2 \setminus S$ are among the basic properties of the integral by which Φ_S is defined. The estimate in (a) follows immediately from the definition of the kernel function.

The properties (a) and (b) of the lemma are verified so far only for rectifiable arcs. They are extended to any B-regular set as in the proof of Lemma 3.2. The assertion (c) for a straight line segment is elementary and can be extended to any linear B-regular set again as in the proof of Lemma 3.2. □

Lemma 4.2 *For* $j = 1, 2$ *let* $E_j \in \mathrm{Bor}\,(\mathbb{R}^2)$ *with* $|E_j|_1 < \infty$ *and* $\mathfrak{R}(E_j)$ *strongly rectifiable. Let* $T = \{z : C_{E_1}(z) = C_{E_2}(z)\}$. *Then the following are equivalent:*

($\bullet$) $|\mathbb{R}^2 \setminus T|_2 = 0$.

($\bullet\bullet$) *T is metrically dense in* $\mathbb{R}^2$.

PROOF. We shall show that ($\bullet\bullet$) implies ($\bullet$), the converse being obvious. By Lemma 4.1 (a) both functions $\Phi_{\mathfrak{R}(E_j)}$ are continuous on the open set $\mathbb{R}^2 \setminus cl(\mathfrak{R}(E_1) \cup \mathfrak{R}(E_2))$ which due to strong rectifiability of $\mathfrak{R}(E_j)$ is dense and of full $|\cdot|_2$ measure in $\mathbb{R}^2$. Furthermore, ($\bullet\bullet$) and Lemma 4.1 (b) imply that $C_{\mathfrak{R}(E_1)} = C_{\mathfrak{R}(E_2)}$ on that set. Hence the equality on it of the two cross integrals, $C_{\mathfrak{R}(E_j)}$ on a set of $|\cdot|_2$ full measure in $\mathbb{R}^2$. By passing to the full sets E_j we note that by Lemma 3.4 (a) the equality might be further affected by the B-irregular parts only on a set of $|\cdot|_2$ measure zero. Thus (•) follows. □

The following lemma deals with the properties of the operators defined in the previous section acting on Φ_E.

Lemma 4.3 *Let $E \in \mathrm{Bor}\,(\mathbb{R}^2)$ and let $z \in \mathbb{R}^2$. Suppose $\delta(z, E) > 0$.*

(a) *Then $D_\nu \Phi_E(z)$ is well defined for $\nu \in C$. Moreover* $D_\nu \Phi_E(z) = \int_E D_\nu \phi\,(u - z, \ell(E, u))\ |du|_1$.

(b) *For $\ell = \ell^0$ we have $S_\ell \Phi_E(z) = \int_{Ts\,(E,z)} \dfrac{\cos(\angle[\ell^0, \ell(E,u)])}{|u-z|^2}\,|du|_1$. This expression vanishes for $z \notin Sp(E)$ and is positive for $z \in Sp(E)$.*

(c) *$\overline{S}_\ell \Phi_E(z)$ for $\ell \in L$ coincides with $S_\ell \Phi_E(z)$ on $\{z : \mathbb{R}^2 \setminus cl(E)\}$. It is also well defined for $z \in cl(E)$ and for those lines $\ell(z, \nu)$ where the limits $\Phi_E(z \pm 0\nu) = \lim_{t\to 0} \Phi_E(z \pm t\nu)$ exist.*

(d) *We have $\overline{S}_{\ell^0} \Phi_E(z) \geq 0$ and $\overline{S}_\ell \Phi_E = 0$ except for the points of the spectrum of E.*

(e)
$$|D_\nu \Phi_E(z)| \leq \frac{|E|_1}{\delta(z, E)^2}.$$

PROOF. (a) is obtained by interchanging differentiation and integration which is valid since by Lemma 2.3 (c) and (d) the integrand is bounded and is a Lipschitz function in u outside of a certain neighborhood of z.

(b) From formula (a) of this lemma with ν replaced by $-\nu$ and adding both expressions one gets $2 \int_{\mathrm{Ts}\,(E,z)} S_\ell \phi\,(u - z, \ell(E,u))\ |du|_1$. Observe that the S_ℓ operator applied at a point of differentiability of function yields zero and apply Lemma 2.3 (f). Finally note that for $z \in \mathbb{R}^2 \setminus \mathrm{cl}\,(E)$ and $z \notin \mathrm{Sp}\,(E)$ the integration is over a set of $|\cdot|_1$ measure zero; otherwise over a set of positive measure.

(c) By (a) of this lemma and by the continuity of Φ_E at z we have for $\ell = \ell(z, \nu)$

$$\overline{S}_\ell \Phi_E(z) = \limsup_{t\to 0+} \frac{\Phi_E(z + t\nu) + \Phi_E(z - t\nu) - 2\Phi_E(z)}{t} =$$

$$\lim_{t\to 0+} \frac{\Phi_E(z + t\nu) - \Phi_E(z) + \Phi_E(z - t\nu) - \Phi_E(z)}{t} = S_\ell \Phi_E(z).$$

(d) On $\mathbb{R}^2 \setminus \mathrm{cl}\,(E)$ the function Φ_E is continuous. Hence $\overline{S}_\ell \Phi_E(z) = S_\ell \Phi_E(z)$ there. Thus its non-negativity follows from (b) of this lemma. The expression is zero except for the points of $Sp(E)$ which may be (within the complement of E) either points of $\cup\{\ell : |\ell \cap E|_1 > 0\}$ or the points of the set difference $\mathrm{Sp}(E) \setminus \cup\{\ell : |\ell \cap E|_1 > 0\}$.

Clearly, if z and z' are two distinct points of the latter, and ℓ is the line joining them, then $|\ell \cap E|_1 = 0$. Hence $\mathrm{Ts}\,(E, z)$ and $\mathrm{Ts}\,(E, z')$ are almost disjoint. Consequently, the set difference additional set is at most countable.

(e) This estimate follows from (a) of this lemma and from the estimate in Lemma 2.3 (c)

5. Sweep and Induced Mass

Definition 11 *Let E and A be two disjoint Borel sets in $\mathbb{R}^2$ with E rectifiable. Then the sweep through A from E, $\sigma(A, E)$,is defined by* $\sigma(A, E) = \int_E |du|_1 \int_{\ell(E,u)} \frac{2\chi_A(z)}{|u - z|^2}\, |dz|_1.$

Clearly the sweep is nonnegative, countably additive in both its set arguments and monotonic; that is, $\sigma(A, E) \le \sigma(A', E')$ when $A \subset A'$ and $E \subset E'$.

Definition 12 *Let F be a function with $\mathrm{dom}\,F$ open, let $G \subset \mathrm{dom}\,F$ be a regular multidomain with $\delta(G, \mathbb{R}^2 \setminus \mathrm{dom}\,F) > 0$ and let $\nu(z)$ denote the interior unit normal to ∂G which is well defined (except for at most finitely many points) unit-vector valued function for $z \in \partial G$. Then the $F-$ induced mass in G, $\Gamma(G, F)$, is defined by $\Gamma(G, F) = \oint_{\partial G} D_{\nu(z)} F(z)\, dz$.*

In particular when F is the full angle of visibility Φ_E of a rectifiable set E with $\delta(E, \mathrm{dom}\,\Phi_E) > 0$, we write $\Gamma(G, E)$ for $\Gamma(G, \Phi_E)$. Let $E \in \mathrm{Bor}\,(\mathbb{R}^2)$ be rectifiable with $|E|_1 < \infty$ such that $E = \mathfrak{R}\,(E)$. When $F = C_E$, by Definition 5, (i) and by Lemmas 3.4 (a) and 4.1 (b) $D_\nu C_E(z) = D_\nu \Phi_{\mathfrak{R}\,(E)}(z)$ for $z \in G$, and, consequently, $\Gamma(G, C_E) = \Gamma(G, \mathfrak{R}\,(E))$.

For the time being we shall concentrate on establishing a few properties of the sweep.

Lemma 5.1 *Let $\ell \in L$ and let $E \subset \ell$ be rectifiable.*

(a) Then $\sigma(A, E) = \int_\ell S_{\ell^\perp} \Phi_E(z) \chi_A(z)\, |dz|_1$.

(b) Let $z \in E$ and $r > 0$. Suppose $0 < |(U(z, r) \cap \ell(E, z) \setminus E)|_1 < 2r$. Then $\sigma((U(z, r) \setminus E), E) = \infty$.

PROOF. (a) Changing the order of integration in the expression defining the sweep we get $\sigma(A,E) = \int_\ell \chi_A(z)\,|dz|_1 \int_\ell \frac{2\chi_E(u)}{|u-z|^2}\,|du|_1$. In the case under consideration $\mathrm{Ts}\,(E,z) = E$ and (a) is obtained by substituting for the inner integral in u its value from Lemma 4.3 (b).

(b) It suffices to let $J \subset \mathbb{R}$ be an interval and let $E \subset \mathbb{R}$ be measurable such that $0 < |J \cap E|_1 < |J|_1$. Show $\sigma((J \setminus E), E) = \infty$. In this setting we have

$$\sigma((J \setminus E), E) = \int_{J\setminus E} dx \int_E \frac{dy}{(x-y)^2} = \int_{(J\setminus E)\times E} \frac{dxdy}{(x-y)^2}.$$

Observe that the product $(J \setminus E) \times E$ intersected with the diagonal line $y = x - t$, $t \in \mathbb{R}$ yields $\{x : x \in J \setminus E, y \in E, y = x - t\} = (J \setminus E) \cap (E + t)$. By the change of coordinates $x = x, t = x - y$

$$\int_{(J\setminus E)\times E} \frac{dxdy}{(x-y)^2} = \int_{-\infty}^{\infty} \frac{|(J \setminus E) \cap (E+t)|_1\,dt}{t^2}.$$

Observe again that

$$J \cap ((E+t) \setminus E) = J \cap (E+t) \setminus J \cap E = (J \setminus E) \cap (E+t).$$

Thus by Lemma 3.2 for $h > 0$ sufficiently small $|(J \setminus E) \cap (E+t)|_1 > t/2$ for $0 \le t < h$ as a result of which $\int_{-\infty}^{\infty} \frac{|(J \setminus E) \cap (E+t))\,|_1 dt}{t^2} \ge \frac{1}{2}\int_{-\epsilon}^{\epsilon} \frac{dt}{t} = \infty$.
□

Lemma 5.2 *Let E be rectifiable, let $z \in cl(E) \setminus SLI(E))$ and let $r > 0$. Then $\sigma((U(z,r) \cap \ell(E,z) \setminus E), E) = \infty$.*

PROOF. Under the assumption, for $E \cap U(z,r)$ at least one of the following three situations occurs

(a) $|U(z,r) \cap NL(E)|_1 > 0$.

(b) $U(z,r) \cap WL(E) \ne \emptyset$.

(c) $U(z,r) \cap \mathrm{SLN}\,(E) \ne \emptyset$.

The proof deals with each of them in order.

(a) Set $U = U(z,r)$. For $u \in U \cap NL(E)$ we have $\int_{\ell(E,u)} \frac{\chi_U(z')}{|z'-u|^2}\,|dz'|_1 = \infty$. Integration of the above in u over $U \cap NL(E)$ obviously produces infinity.

(b) Let $z' \in U(z,r) \cap WL(E)$ and let $U(z',r') \subset U(z,r) \setminus \ell(E,z)$ where $r' > 0$. By Lemma 5.1 (b), $\sigma((U(z',r') \cap \ell(E,z') \setminus E), E) = \infty$ from which by monotonicity of the sweep the conclusion follows.

(c) Let $z_0 \in U(z,r) capsln(E)$. The assumption of not being metrically laterally isolated means that z_0 is in the closure of that part of E of positive linear measure which is outside $\ell(E, z_0)$, which means z_0 is the limit of a sequence, $z_n \in E \setminus \ell(E, z_0)$, each of metric density of E. For every $n \in \mathbb{N}$ and for $s_n > 0$ arbitrarily small $|U_n \cap E|_1 > 0$, where $U_n = U(z_n, s_n)$. For n large enough $U_n \subset U(z,r)$. If $z_n \in WL(E) \cup NL(E)$, then (c) holds already by (a) and (b) of this lemma. Thus consider only the case $z_n \in SL(E) \setminus \ell(E, z_0)$. (The tangents $\ell(E, z_n)$ may be assumed to exist.) By rectifiability of E, we have $\lim_{n\to\infty} |\ell(E, z_n) \cap E|_1 = 0$. This means that there are numbers $r_n > 0$ with $\lim_{n\to\infty} r_n = 0$, such that $|(U(z_n, r_n) \cap \ell(E, z_n)) \setminus E|_1 > 0$ for n sufficiently large. By Lemma 5.1 (b) this implies $\sigma((U(z_n, r_n) \cap E \setminus \ell(E, z_n)), E) = \infty$ and the presence of even one $U(z_n, r_n)$ within $U(z,r)$ leads again to the desired conclusion.

6. Multilinear Sets Inscribed in a Measurable Subset of a Rectifiable Arc.

Let S be a rectifiable arc and let $\zeta = \{z_0, ..., z_m\}$, where $z_k \in S$, partition S into subarcs. Inscribe in S the polygon S_ζ using the ζ partition points as vertices and define a mapping $p_\zeta : S \to S_\zeta$ by projecting every point $z \in S$ orthogonally onto the chord connecting the two partition points between which z lies on S. (The vertices map onto themselves.) For a Borel set $E \subset S$ the multilinear set $p_\zeta(E)$ is denoted by E_ζ. This set has at every point (except for the vertices and isolated points of the set) a well defined tangent line which remains constant along each linear component. Use $v = p_\zeta(u)$ to reparametrize the integral formula for Φ_{E_ζ} to the set E after which it takes the form

$$\Phi_{E_\zeta}(z) = \int_{E_\zeta} \phi(v - z, \ell(E_\zeta, v))\, |dv|_1 = \int_E \phi(p_\zeta(u) - z, \ell(E_\zeta, p_\zeta(u)) \frac{dp_\zeta(u)}{du}\, |du|_1.$$

Here the factor $\dfrac{dp_\zeta(u)}{du}$ in the integrand has a simple geometric meaning. It equals $\cos \angle[\ell(E,u), \ell(E_\zeta, p(u)]$.

Let $\{\zeta(n)\}_{n=1}^{\infty}$ be a sequence of partitions of S of increasing refinement. Set $p_n = p_{\zeta(n)}$ and $E_n = p_{\zeta(n)}(E)$. Set $\ell(u) = \ell(E,u)$ and $\ell_n(u) = \ell(E_n, p_n(u))$. The above formula becomes

$$\Phi_{E_n}(z) = \int_E \phi(p_n(u) - z, \ell_n(u) \cos \angle[\ell(u), \ell_n(u)]\, |du|_1. \tag{6*}$$

Lemma 6.1 (a) *For every* $u \in E$ *we have* $\lim_{n\to\infty} p_n(u) = u$. *For* $|\cdot|_1$ *almost every* $u \in E$ *we have* $\lim_{n\to\infty} \angle[\ell_n(u), \ell(u)] = 0$. *For* $z \notin Sp(E)$ *with*

$\delta(z, E) > 0$ *we have*

$$\lim_{n\to\infty} \phi(p_n(u) - z, \ell_n(u)) \cos \angle[\ell(u), \ell_n(u))] = \phi(u - z, \ell(u)) \qquad (6.1.a*)$$

and

$$\lim_{n\to\infty} D_\nu(\phi(p_n(u) - z, \ell_n(u)) \cos \angle[\ell(u), \ell_n(u)]) = D_\nu\phi(u - z, \ell(u)). \quad (6.1.a**)$$

(b) *For G a regular multidomain with $\delta(G, E) > 0$ we have* $\lim_{n\to\infty} \sigma(G, E_n) = \sigma(G, E)$.

(c) *For $\delta(z, E) > 0$ we have $\lim_{n\to\infty} \Phi_{E_n}(z) = \Phi_E(z)$.*

(d) *For $\delta(z, E) > 0$, $\nu \in C$ and $z \notin Sp(E)$ we have* $\lim_{n\to\infty} D_\nu \Phi_{E_n}(z) = D_\nu \Phi_E(z)$.

(e) *For G a regular multidomain with $\delta(G, E) > 0$ we have* $\lim_{n\to\infty} \Gamma(G, E_n) = \Gamma(G, E)$.

PROOF. (a) The first limit relationship is the consequence of the definition of the p_n mappings. Since E is rectifiable, $\ell(u)$ exists at $|\cdot|_1$ almost every point $u \in E$. Let u be such a point. The tangent $\ell(E_n, v)$, $v \in E_n$ represent as a function $\ell(E_n, p_n(u))$ in the variable u, defined over E. Considering that $\lim_{n\to\infty} p_n(u) = u$ and the fact that as $n \to \infty$ the line carrying the chord containing $p_n(u)$ converges to $\ell(E, u)$ in set theoretical sense, the equation (6.1.a*) follows from the continuity of the ϕ function in both of its arguments. Likewise, (6.1.a**) follows from Lemma 2.3 (d)

(b) As a result of (a) of this lemma and in view of the assumed regularity of the boundary, ∂G, we have $\lim_{n\to\infty} G \cap \ell(E_n, p_n(u)) = G \cap \ell(u)$ in the set-theoretical sense a.e. on E and $\lim_{n\to\infty} |G \cap \ell(E_n, p_n(u))|_1 = |G \cap \ell(u)|_1$. But then $\lim_{n\to\infty} \int_{\ell_n(u)} \frac{\chi_G(w)}{|w - p_n(u)|^2} |dw|_1 = \int_{\ell(u)} \frac{\chi_G(w)}{|w - u|^2} |dw|_1$ a.e. on E. Since $\inf_n \delta(G, E_n) > 0$, the integrands are uniformly bounded on E. Thus integration of this sequence over E proves (b).

(c) The integrands in (6*) are all bounded in u uniformly in n for n large enough (depending on z) which in view of (a) of this lemma justifies passage to limit.

(d) For E and E_n use the integral representation for Φ functions offered by Lemma 4.3 (a). When the $\ell(E_n, v)$, $v \in E_n$ has been represented as $\ell(E_n, p_n(u))$ over E the representation of Φ takes the form

$$D_\nu \Phi_{E_n}(z) = \int_E |D_\nu(\phi(p_n(u) - z, \ell_n(u) \cos \angle[\ell(u), \ell_n(u)]) \, du \qquad (6.1.d*).$$

For a z subject to conditions indicated we have $|\mathrm{Ts}\,(E, z)|_1 = 0$ and, as follows from Lemma 2.3 (d) estimate (2.3d**), the integrand in (6.1.d*) is bounded in the variable u uniformly in n for n sufficiently large. This together with (6.1.a**) of this lemma justifies passage to limit as $n \to \infty$ in (6.1.d*).

(e) From the definition of G it follows immediately that $\partial G \cap \mathrm{Sp}\,(E)$ is at most countable. Thus (e) is a consequence of (d) of this lemma taking into account the inequality of Lemma 4.3 (e) by which the integrands in the integral representation of $\Gamma(G, E_n)$ are uniformly bounded. □

Lemma 6.2 *Let $E \in \mathrm{Bor}\,(\mathbb{R}^2)$ be rectifiable and let G be a regular multidomain with $\delta(G, E) > 0$. Then $\Gamma(G, E) = -\sigma(G, E)$.*

PROOF. In view of the results of Lemma 6.1, it suffices to show that this equality holds for a multilinear set. Let E be such a set and let $E^k = \ell^k \cap E$, $k = 1, ..., m$ be its linear components. Let G^k_s, $s = 0, 1$; $k = 1, ..., m$ be the two multidomains into which ℓ^k splits G, one on either side of ℓ^k (G^k_0 or G^k_1 could be empty). Their boundaries are $\partial G^k_s = (\ell^k \cap G) \cup \partial(G^k_s \setminus \ell^k)$. For a given k we have $\partial G = (\partial G^k_0 \setminus \ell^k) \cup (\partial G^k_1 \setminus \ell^k)$. Accordingly, writing $\nu(w)$ for the interior unit normal vector to a respective boundary at w,

$$\Gamma(G, E^k) = \left(\int_{\partial G^k_0 \setminus \ell^k} + \int_{\partial G^k_1 \setminus \ell^k} \right) D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1 =$$

$$\left(\oint_{\partial G^k_0} + \oint_{\partial G^k_1} \right) D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1 - 2 \int_{G \cap \ell^k} D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1.$$

Since neither of the two multidomains G^k_s, $s = 0, 1$ has points of $\mathrm{Sp}(E^k) = \ell^k$ in its interior, $\Phi_{E^k}(z)$ is harmonic in the interior of each of them. Thus $\oint_{\partial G^k_0} D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1 = 0$ and $\oint_{\partial G^k_1} D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1 = 0$. Hence

$$\Gamma(G^k, E^k) = -2 \int_{G \cap \ell^k} D_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1 = - \int_{G \cap \ell^k} S_{\nu(w)} \Phi_{E^k}(w)\, |dw|_1$$

for $s = 0, 1$. Application of Lemma 5.1 (a) to the right-hand expression yields $\Gamma(G, E^k) = -\sigma(G, E^k)$. Thus the lemma holds for E^k. By additivity

$$\Gamma(G, E) = \sum_{k=1}^{n} \Gamma(G, E^k) = -\sum_{k=1}^{n} \sigma(G, E^k) = -\sigma(G, E). \qquad \square$$

[Note: The equality of Lemma 6.2 shows that induced mass is nonpositive.]

7. The Δ and Δ' Point-indices

Definition 13 *Let F be a function with an open domain, $dom F$, and let $z \in cl(dom F)$. The Δ-point index of F at z, $\Delta(F,z)$, is defined to be*

$$\lim_{r\to 0} \inf\{\Gamma(G,F) : G \textit{ is a regular multidomain with } G \subset U(z,r) \cap dom F\}.$$

If $\operatorname{dom} F$ is dense in $\mathbb{R}^2$, then $\Delta(F,z)$ is well defined for every $z \in \mathbb{R}^2$. By the equality of Lemma 6.2 and by the monotonicity of $\sigma(A,E)$ in its first set-argument

$$\inf\{\Gamma(G,F) : G \subset U(z,r) \cap \operatorname{dom} F\} = -\sigma((U(z,r) \cap \operatorname{dom} F), E).$$

For F take Φ_E where E is a strongly rectifiable set. The domain in this case is an open dense subset of $\mathbb{R}^2$ and the right-hand side of the above expression becomes $-\sigma((U(z,r) \setminus E), E)$. Moreover $\Delta(\Phi_E, z)$ is well defined on $\mathbb{R}^2$ and $\Delta(\Phi_E, z) = -\lim_{r\to 0} \sigma((U(z,r) \setminus E), E)$. We accept $-\infty$ as a possible value.

Definition 14 *Let F be a function with $dom F$ open and let $z \in cl(dom F)$. The Δ'-point index of F at z, $\Delta'(F,z)$, is defined by*

$$\Delta'(F,z) = \inf\{\overline{S}_\ell F(z) : \ell \in L^0\}.$$

Lemma 7.1 *Let E be rectifiable. Then*

$$\Delta(\Phi_E, z) = \begin{cases} -\infty & \textit{if } z \in cl(E \setminus SLI(E)) \\ 0 & \textit{if } z \in SLI(E). \end{cases}$$

PROOF. For $z \in \operatorname{cl}(E \setminus \operatorname{SLI}(E))$ and $r > 0$ Lemma 5.2 implies

$$\inf\{\Gamma(G,\Phi) : G \subset U(z,r) \cap \operatorname{dom} \Phi\} = -\infty.$$

Thus $-\infty$ is the limit of this expression as $r \to 0$. On the other hand, when $z \in \operatorname{SLI}(E)$, there is a $r > 0$ for which $U(z,r)$ is essentially cut by a linear part of E into two half disks. For a multidomain G with $G \subset U(z,r) \setminus E$ we have $\sigma(U(z,r) \setminus E, E) = 0$ since $\ell(E,z') \cap G = \emptyset$ for $z' \in U(z,r) \cap E$, which completes the proof of the lemma.

Lemma 7.2 *Let $E \in \operatorname{Bor}(\mathbb{R}^2)$ be rectifiable and let $\delta(z,E) > 0$. Then*

(a) $\Delta(\Phi_E, z) = 0$

(b) $\Delta'(\Phi_E, z) \geq 0$

PROOF. (a) The r infimum appearing in the definition of $\Delta(\Phi_E, z)$ is realized for the closed neighborhood $\operatorname{cl} U(z,r)$ (which is a regular multidomain.) Integrating the inequality of Lemma 4.3 (e) in $|\cdot|_1$ over ∂G one obtains $|\Gamma(G,E)| \leq \dfrac{|\partial G|_1 |E|_1}{\delta(G,E)}$ from which $\Delta(\Phi_E, z) = 0$ follows.

(b) By Lemma 4.3 (b), (c) $\overline{D}_\ell \Phi_E(z) = 0$ for those $z \notin \mathrm{Sp}(E)$ and for any $\ell \in L^0$. For $z \in \mathrm{Sp}(E)$ the value may be positive. □

8. Boundary Behavior of Φ_E on SLI

Since this case is treated by methods different from the rest, this section is devoted to it.

Lemma 8.1 *Let E be closed and rectifiable and let $z_0 \in SLI(E)$. There is a $r > 0$ such that on a dense set of points $z \in U(z_0, r) \cap E$, $\Delta'(\Phi_E, z) < 0$.*

PROOF. The point z_0 being laterally isolated and E being closed imply that there is a $r > 0$ such that $|U(z_0,r) \cap (E \setminus \ell(E,z_0))|_1 = 0$; that is, every point in $U(z_0,r) \cap E$ is at a positive distance from the metrically dense-in-itself part of $E \setminus \ell(E,z_0)$. Moreover, $U(z_0,r) \cap (E \cap \ell(E,z_0))$ is a segment of the tangent line $\ell(E,z_0)$. $\mathrm{Sp}\,(E) \cap \ell$ is at most countable for each $\ell \not\subset \mathrm{Sp}\,(E)$; in particular, with $Sp(E) \cap \ell(E,z_0)$ is at most countable. This means that on $\ell(E,z_0)$ there is a dense subset of points not from that spectrum. Let $z \in U(z_0,r) \cap E \cap \ell(E,z_0)$ be one of such points.

Partitioning E into $E \cap \ell(E,z_0)$ and $E \setminus \ell(E,z_0)$ we write

$$\Phi_E = \Phi_{E \cap \ell(E,z_0)} + \Phi_{E \setminus \ell(E,z_0)}.$$

According to Lemma 4.3 (d) and our choice of z, $S_{\ell^\perp} \Phi_{E \setminus \ell(E,z_0)}(z) = 0$. Now set $E \cap \ell(E,z_0)$ being closed and rectifiable, there is an interval J such that $J \subset \ell(E,z_0) \setminus E \setminus U(z_0,r)$. Thus for $t \neq 0$ and ν perpendicular to $\ell(E,z_0)$

$$\Phi_J(z \pm t\nu) = \left(\arctan \frac{d + |J|_1}{t} - \arctan \frac{d}{t}\right)$$

$$\Phi_{\ell(E,z_0) \setminus E}(z \pm t\nu) < \Phi_{\ell(E,z_0) \setminus U(z_0,r)}(z \pm t\nu),$$

where $d = \delta(z,J)$. (The inequalities are strong since J is only one of such intervals. Note also that a Φ function of a subset of $\ell(E,z_0)$ is symmetric with respect to this line.) Since

$$\Phi_{\ell(E,z_0) \setminus U(z_0,r)}(z \pm 0\nu) = \Phi_{\ell(E,z_0) - E}(z \pm 0\nu) = 0, \tag{8.1*}$$

we have

$$\liminf_{t\to 0+} 2\frac{\Phi_{\ell(E,z_0)\setminus E}(z+t\nu)}{t} \geq \lim_{t\to 0+} 2\frac{\Phi_J(z+t\nu)}{t} = S_{\ell^\perp}\Phi_J(z). \qquad (8.1**)$$

The expression on the right side of (8.1**) is positive. Indeed,

$$D_\nu \Phi_J(z\pm t\nu) = -\frac{d+|J|_1}{(d+|J|_1)^2+t^2} + \frac{d}{d^2+t^2}$$

which is continuous in t for $t \neq 0$, right and left continuous at $t=0$. Furthermore $S_{\ell^\perp}\Phi_J(z) = -\dfrac{2}{d+|J|_1} + \dfrac{2}{d} > 0$. Thus $(8.1**)$ leads to

$$\liminf_{t\to 0+} 2\frac{\Phi_{\ell(E,z_0)\setminus E}(z+t\nu)}{t} > 0. \qquad (8.1***)$$

Observe now that for $z \notin \ell(E,z_0)$ we have $\Phi_{\ell(E,z_0)}(z) = \pi = \Phi_{E\cap\ell(E,z_0)}(z) + \Phi_{\ell(E,z_0)\setminus E}(z)$. As a result of (8.1*), (8.1***) and of the last equality $\Phi_{E\cap\ell(E,z_0)}(z\pm 0\nu) = \pi$ and

$$\overline{S}_{\ell^\perp}\Phi_{E\cap\ell(E,z_0)}(z) = \limsup_{t\to 0+} 2\frac{\Phi_{E\cap\ell(E,z_0)}(z+t\nu)-\pi}{t} =$$

$$-\liminf_{t\to 0+} 2\frac{\Phi_{\ell(E,z_0)\setminus E}(z+t\nu)}{t} < 0.$$

It follows that for an arbitrary $\ell = \ell(z,\nu)$ $\overline{S}_\ell\Phi_E(z) < 0$, which completes the proof of this lemma. □

9. The 'Probing Touch' Index $\oplus$

Definition 15 *Let F be a function with dom$F$ open and let $z \in cl(domF)$. Then*

$$\oplus(F,z) = \min\{\Delta(F,z), \Delta'(F,z)\}$$

(with $-\infty$ as an accepted value).

Lemma 9.1 *Let $E \in$ Bor$(\mathbb{R}^2)$ with $|E|_1 < \infty$ and with strongly rectifiable $\mathfrak{R}(E)$. Then*

$$\oplus(C_E,z) = \begin{cases} -\infty & \text{if } z \in cl(\mathfrak{R}(E)) \\ 0 & \text{if } z \notin cl(\mathfrak{R}(E)). \end{cases}$$

PROOF. It follows from Lemmas 7.1, 7.2 and 8.1 that when $z \in$ cl$(\mathfrak{R}(E))$ either $\Delta(C_E,z) = -\infty$ or $\Delta'(C_E,z) < 0$ and for $z \notin$ cl$(\mathfrak{R}(E))$ both Δ and Δ' indices vanish. □

Its ability to localize the essential proximity of a point to a set under consideration justifies application of the designation 'probing touch' to this index.

10. Conclusion

We are now able to present the major results of this paper.

Theorem 10.1 (Crofton weak transform inversion) *Let* $E \in \mathrm{Bor}(\mathbb{R}^2)$ *with* $|E|_1 < \infty$ *and with strongly rectifiable* $\mathfrak{R}(E)$. *Then*

$$cl(\mathfrak{R}(E)) = \{z : \oplus(C_E, z) < 0\}.$$

The assertion follows directly from the Lemma 9.1.

Theorem 10.2 *Let* $E_j\ j = 1, 2$ *be two Borel subsets of* $\mathbb{R}^2$ *with strongly rectifiable B-regular parts whose cross integral functions agree on a metrically dense subset of* $\mathbb{R}^2$. *Then*

$$cl(\mathfrak{R}(E_1) = cl(\mathfrak{R}(E_2).$$

PROOF. Under the assumption made by the Lemma 4.3 the cross integrals of both sets are $|\cdot|_2$ equivalent. Thus according to Theorem 10.1 the closures of the B-regular parts of both sets being determined by their cross integrals, are identical.

Acknowledgment The author expresses his gratitude to Professor Pertti Mattila who read this paper in manuscript form and made numerous valuable suggestions and remarks. In addition the author wishes to express his love and gratitude to his wife, Nora, for her devotion and moral support during the preparation of this article.

References

[1] E. W. Hobson, *The Theory of Functions of a Real Variable*, Dover Publications Inc. New York,1927.

[2] A. S. Besicovitch, *On the fundamental geometric properties of linearly measurable plane sets of points (II)*, Math Ann., **115** (1938), 296–329.

[3] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York Inc.1969.

[4] H. Fast and A. Goetz, *Sur l'integrabilite riemanniennė de la function de Crofton*, Ann. Soc. Polon. Math., **25** (1952), 309–322.