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Haseo Ki* and Tom Linton, Department of Mathematics, Caltech, Pasadena CA 91125

NORMAL NUMBERS AND SUBSETS OF $\mathbb{N}$ WITH GIVEN DENSITIES

For a set X, we would like to find the exact level of X in the Borel or difference hierarchy. In addition, look for "natural" arising sets (like $\mathbb{Q}$, C^∞ etc) that are non-ambiguous sets of high levels. In an attempt to prove that the set of real numbers which are normal to at least one base $n \in \mathbb{N}$, is $\mathbf{\Sigma}^0_4$ non $\mathbf{\Pi}^0_4$, we were naturally lead to the problem of classifying the Borel class of subsets of $\mathbb{N}$ whose densities lie in $X \subseteq [0,1]$, in terms of the Borel class of X. This work produced several new natural examples of nonambiguous Borel sets and answered two questions of Kechris.

For Polish topological spaces X, let $\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ denote the Borel sets of additive class α, $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ the sets of multiplicative class α, and $\mathbf{\Delta}^0_{\boldsymbol{\alpha}} = \mathbf{\Sigma}^0_{\boldsymbol{\alpha}} \cap \mathbf{\Pi}^0_{\boldsymbol{\alpha}}$, be the ambiguous sets of class α. Thus, $\mathbf{\Sigma}^0_1 =$ Open, $\mathbf{\Pi}^0_1 =$ Closed, $\mathbf{\Sigma}^0_2 = F_\sigma$, $\mathbf{\Pi}^0_2 = G_\delta$, and so on. More generally we can define the difference hierarchy which is a finer two sided hierarchy on the $\mathbf{\Delta}^0_{\boldsymbol{\alpha}}$ sets. It extends the Borel hierarchy by including it as the first level ($\xi = 1$) for each countable ordinal α. For each countable ordinal $\xi \geq 1$, let $\mathcal{D}_\xi(\mathbf{\Pi}^0_{\boldsymbol{\alpha}})$ denote the sets which are nested differences of ξ many $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$ sets. So $\mathcal{D}_1(\mathbf{\Pi}^0_{\boldsymbol{\alpha}}) = \mathbf{\Pi}^0_{\boldsymbol{\alpha}}$, $\mathcal{D}_2(\mathbf{\Pi}^0_{\boldsymbol{\alpha}}) = \{\, A - B \mid A, B \in \mathbf{\Pi}^0_{\boldsymbol{\alpha}}, \text{ and } A \supseteq B \,\}$ and $\mathcal{D}_3(\mathbf{\Pi}^0_{\boldsymbol{\alpha}})$ is the collection of sets of the form

$$(A - B) \cup C \text{ where } A, B, C \in \mathbf{\Pi}^0_{\boldsymbol{\alpha}} \text{ and } A \supseteq B \supseteq C.$$

For any class of sets Γ, let the dual class, $\widetilde{\Gamma}$, be the collection of complements of sets in Γ, and say A is properly Γ, if $A \in \Gamma - \widetilde{\Gamma}$.

Definition 1 *For $A \subseteq \mathbb{N}$, let $\delta(A) = \lim\limits_{n\to\infty} \dfrac{|A \cap [0,n)|}{n}$, if the limit exists, and say $\delta(A)$ does not exist, otherwise. We call $\delta(A)$ the density of A.*

For nonempty $X \subseteq [0,1] \cap \mathbb{R}$, let $D_X = \{\, A \subseteq \mathbb{N} \mid \delta(A) \in X \,\}$. If we identify $A \subseteq \mathbb{N}$ with its characteristic function, then D_X becomes a subset of the Cantor space $2^{\mathbb{N}}$ (with the usual product topology).

*Presenter

As limit is a $\mathbf{\Pi}^0_3$ notion, the Borel class of D_X should jump well above that of X, surprisingly the jump is almost exactly one level, and a direct relationship between the exact Borel class of X and the exact Borel class of D_X can be exhibited.

Theorem 1 *Let $X \subseteq [0,1]$ be nonempty.*

X	D_X
i) $\mathbf{\Pi}^0_2$ *or simpler*	*properly* $\mathbf{\Pi}^0_3$
ii) properly $\mathbf{\Sigma}^0_2$	*properly* $\mathcal{D}_2(\mathbf{\Pi}^0_3) = \mathbf{\Pi}^0_3 \cap \mathbf{\Sigma}^0_3$
iii) properly $\mathbf{\Pi}^0_{\boldsymbol{n}}(\mathbf{\Sigma}^0_{\boldsymbol{n}})$ $(3 \leq n < \omega)$	*properly* $\mathbf{\Pi}^0_{\boldsymbol{n}+1}(\mathbf{\Sigma}^0_{\boldsymbol{n}+1})$
iv) properly $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}(\mathbf{\Sigma}^0_{\boldsymbol{\alpha}})$ $(\alpha \geq \omega)$	*properly* $\mathbf{\Pi}^0_{\boldsymbol{\alpha}}(\mathbf{\Sigma}^0_{\boldsymbol{\alpha}})$
v) properly $\mathcal{D}_\xi(\mathbf{\Pi}^0_{\boldsymbol{\alpha}})$ $(\alpha \geq 2)$	*properly* $\mathcal{D}_\xi(\mathbf{\Pi}^0_{\mathbf{1}+\boldsymbol{\alpha}})$ $(1+\alpha = \alpha$ *if* $\alpha \geq \omega)$
vi) properly $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_{\boldsymbol{\alpha}})$ $(\alpha \geq 3$ *or* $[\alpha = 2$ *and* $\xi \geq \omega])$	*properly* $\widetilde{\mathcal{D}}_\xi(\mathbf{\Pi}^0_{\mathbf{1}+\boldsymbol{\alpha}})$
vii) properly $\widetilde{\mathcal{D}}_m(\mathbf{\Pi}^0_2)$ $(m < \omega)$	*properly* $\widetilde{\mathcal{D}}_{m+1}(\mathbf{\Pi}^0_3)$

Thus except for vii) and i) the class jumps by exactly one level. In particular i) says the subsets of $\mathbb{N}$ with density 0 (or 1) are properly $\mathbf{\Pi}^0_3$ and i) can be used to show $\forall n \geq 2$ the real numbers that are simply normal or normal to base n are properly $\mathbf{\Pi}^0_3$.

ii) shows $D_{\mathbb{Q}}$ is $\mathbf{\Delta}^0_4$ non $\mathbf{\Pi}^0_3$ non $\mathbf{\Sigma}^0_3$. Hence we have a fairly natural set above the third level, but not on the fourth level. We don't know of any "natural" example properly on or above the fourth level.

Conjecture: $S_{\mathbb{N}}$ = The real numbers that are normal to at least one base $n \geq 2$, is $\mathbf{\Sigma}^0_4$ non $\mathbf{\Pi}^0_4$.