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LEBESGUE MEASURE AND GAMBLING

We modify a standard coin flipping game and are thus able to characterize Lebesgue measure on the Cantor space $2^{\mathbb{N}}$, in terms of optimal payoffs in this game. In addition, if either player has a winning strategy in a certain game defined by $P \subseteq 2^{\mathbb{N}}$, then P is measurable.

A typical fair game would have a gambler betting on whether a coin flip will turn up heads or tails (by fair we mean that the probability of heads and tails are both 1/2 and the gambler is paid the amount of their bet when they win). If a gambler plays this game once for each natural number $n \in \mathbb{N}$, the coin flips produce a sequence, $\{p_n\}_{n\in\mathbb{N}}$, of heads and tails, which can be viewed as an element of $2^{\mathbb{N}}$. Our modification introduces a second player who is given the option of selecting the sequence $\{p_n\}_{n\in\mathbb{N}}$, bit by bit, with the requirement that the sequence belongs to a given set $P \subseteq 2^{\mathbb{N}}$. Thus we have two players, G–the gambler and C–the casino. At the start, a payoff set, $P \subseteq \{+1,-1\}^{\mathbb{N}}$, the gamblers initial balance, $B_0 \in \mathbb{R}^+$, and a real number, h, are fixed. The game has one turn for each $n \in \mathbb{N}$ and each turn is played as follows. First, G places a bet b_n, a rational number with absolute value less than or equal to their current balance, say B_n. Seeing this bet, C picks a digit, $p_n = +1$ or -1, and the gambler's balance is updated to

$$B_{n+1} = B_n + p_n \cdot b_n.$$

Hence, a negative bet is a stake that C will play $p_n = -1$ and a positive bet is a stake that C will play $p_n = +1$, since these two situations result in an increase in G's balance. To make the game nontrivial we require that C produces $\{p_n\}_{n\in\mathbb{N}} \in P$. We say that C wins a run of this game iff

$$\sup_{n\in\mathbb{N}} B_n < \frac{B_0}{h}$$

(and neither player "cheats"–the first player to cheat looses and C can cheat by producing $p \notin P$, while G can cheat by betting an irrational amount, or more than his current balance). Call this game $\Gamma(P, B_0, h)$.

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A strategy for one of the players is a rule, which based on the complete history of the game so far, tells the player how much to bet or which digit to play. This notion can be precisely defined in terms of functions. A strategy for either player is a winning strategy if it wins against every strategy of the other player. Intuitively, a winning strategy is a single algorithm which defeats every possible legal play by the opponent.

Let μ denote the $(1/2, 1/2)$ product measure on $2^{\mathbb{N}}$ (i.e. the homogeneous Lebesgue probability measure), μ^+ the corresponding outer measure and μ_- the corresponding inner measure. Let $0 < B_0 < \infty$ be any positive real number and $P \subseteq 2^{\mathbb{N}}$ be nonempty.

Measure zero case: $\mu(P) = 0$ iff G has a winning strategy in $\Gamma(P, B_0, 0)$.

Positive measure G: Let $0 < h \leq 1$. $\mu^+(P) \leq h$ iff $\forall h' > h$, G has a winning strategy in $\Gamma(P, B_0, h')$.

Positive measure C: Let $0 < h \leq 1$. $\mu_-(P) \geq h$ iff for all positive $h' < h$, C has a winning strategy in $\Gamma(P, B_0, h')$.

$\Gamma(P, B_0, h)$ is a standard game of perfect information (on a countable set). Such a game is determined if one of the two players has a winning strategy. For $P \subseteq 2^{\mathbb{N}}$, let

$$h_P = \frac{\mu^+(P) + \mu_-(P)}{2}$$

denote the average of the inner and outer measure of P. The above results show that if $\Gamma(P, 1, h_P)$ is determined, then $\mu^+(P) = \mu_-(P)$ and hence $\mu(P)$ exists.