Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 35–36

Donald W. Spear, 201 E. Mimosa Circle, San Marcos, TX 78666

DIMENSIONS FOR SUBSETS OF $[0,1]$

Define $N_\delta(E) = \min\{k \in N;\ E \subset \bigcup_{j=1}^{k}\{B_\delta(x_j)\}\}$. If $\alpha \geq 0$,

$$\tau_{N,inf}^{\alpha}(E) := \liminf_{\delta\downarrow 0} N_\delta(E)\delta^\alpha \ \textit{and}\ \tau_{N,sup}^{\alpha}(E) := \limsup_{\delta\downarrow 0} N_\delta(E)\delta^\alpha.$$

Set $\mathcal{M}^\alpha(E) = \inf\{\sum_{j=1}^{\infty}\tau_{N,inf}^{\alpha}(A_j) : \{A_j\}_{j=1}^{\infty}\ covers\ E\}$ and

$$\mathcal{N}^\alpha(E) = \inf\{\sum_{j=1}^{\infty}\tau_{N,sup}^{\alpha}(A_j) : \{A_j\}_{j=1}^{\infty}\ covers\ E\}.$$

The modified lower and upper capacity dimensions are

$$dim_{M\underline{C}}E = \inf\{\alpha \geq 0;\ \mathcal{M}^\alpha(E) = 0\}$$

and

$$dim_{M\overline{C}}E = \inf\{\alpha \geq 0;\ \mathcal{N}^\alpha(E) = 0\}.$$

If $\mathcal{H}_{\mathcal{S}}^{\alpha}(E) = \lim_{\delta\downarrow 0}\inf\{\sum_{B_\epsilon(x)\in\mathcal{G}}(2\epsilon)^\alpha;\ \mathcal{G}\ is\ a\ \mathcal{S}\text{–}covering\ of\ E,\ 2\epsilon \leq \delta\}$,

$$dim_{\mathcal{H}}E := \inf\{\alpha \geq 0;\ \mathcal{H}_{\mathcal{S}}^{\alpha}(E) = 0\}.$$

Cajar and Sandau (1985) have shown that given $\gamma, \delta \in R$ with $0 \leq \gamma \leq \delta \leq 1$, there is a construction which produces a perfect set $P \subset R$ with $dim_{\mathcal{H}}P = \gamma,\ dim_{M\underline{C}}P = \delta\ and\ dim_{M\overline{C}}P = 1$.

The set of closed intervals centered in E is $\mathcal{S}_E$. A $\mathcal{S}_E$-packing of E has elements from $\mathcal{S}_E$ that are pairwise disjoint. If

$$P_{\mathcal{S}_E}^{\alpha}(E) = \limsup_{\delta\downarrow 0}\{\sum_{B_\epsilon(x)\in\mathcal{R}}(2\epsilon)^\alpha : \mathcal{R}\ is\ a\ \mathcal{S}_E\text{−}packing\ of\ E,\ 2\epsilon \leq \delta\},$$

$\mathcal{P}_{\mathcal{S}_E}^{\alpha}(E) := \inf\{\sum_{j=1}^{\infty}P_{\mathcal{S}_E}^{\alpha}(A_j) : \{A_j\}_{j=1}^{\infty}\ covers\ E\}$. Taylor calls $\mathcal{P}_{\mathcal{S}_E}^{\alpha}$ the α-packing measure. Each of $\{\mathcal{N}^\alpha;\ \alpha \geq 0\}\ and\ \{\mathcal{P}_{\mathcal{S}_E}^{\alpha};\ \alpha \geq 0\}$ produces $dim_{M\overline{C}}$ which is also called the packing dimension, $dim_{\mathcal{P}}$.

The capacity dimensions are

$$dim_{\underline{C}}E = \inf\{\alpha \geq 0;\ \tau^{\alpha}_{N,inf}(E) = 0\}$$

and

$$dim_{\overline{C}}E = \inf\{\alpha \geq 0;\ \tau^{\alpha}_{N,sup}(E) = 0\}.$$

It can be shown that there is a construction such that given $0 < h < p < v < 1$ and $0 < h < u < v < 1$, a perfect set, X, is produced with $dim_{\mathcal{H}}X = h, dim_{\underline{C}}X = u, dim_{\mathcal{P}}X = p$ *and* $dim_{\overline{C}}X = v$. Taylor has asked if there is a construction for which in addition to the above there is an s such that $h < s < p$, $h < s < u$, $dim_{M\underline{C}} = s$ and for each open set, G, that meets X $dim_{\underline{C}}X \bigcap G = u$ *and* $dim_{\overline{C}}X \bigcap G = v$.