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# RECENT DEVELOPMENTS IN GEOMETRIC INTEGRATION THEORY

This note is an extended summary of my lecture presented at the Macalester Symposium in Real Analysis on June 26, 1993. Detailed proofs of the results stated below can be found in [4, Chapter 12].

While I believe that ultimately the generalized Riemann integral in $\mathbb{R}^m$ ought to be defined by means of BV sets (see [3]), the technical details connected with their use often obscure the main geometric ideas on which the definition rests. Thus it appears useful to show that using only finite unions of intervals, we can still define a viable Rieman-type integral in $\mathbb{R}^m$, whose properties parallel those of the integral based on BV sets.

Throughout, $m$ is a fixed positive integer. The interior and boundary of a set $E \subset \mathbb{R}^m$ is denoted by $E^\circ$ and $\partial E$, respectively. A *cell* is a nondegenerate compact subinterval of $\mathbb{R}^m$. A finite (possibly empty) union of cells is called a *figure*. The diameter $d(A)$, volume $|A|$, and perimeter $\|A\|$ of a figure $A$ are defined in the obvious way. The *regularity* of a figure $A$ is the number

$$r(A) = \frac{|A|}{d(A)\|A\|}$$

if $A \neq \emptyset$, and $r(A) = 0$ otherwise.

All functions we consider are real-valued. With each function $F$ on $\mathbb{R}$ we associate an additive function of cells, and hence of figures, in $\mathbb{R}$ defined by

$$F([a, b]) = F(b) - F(a)$$

for every cell $[a, b]$ in $\mathbb{R}$. We note that $F$ is continuous on $\mathbb{R}$ if and only if the associated functions of figures in $\mathbb{R}$ has the following property: given $\varepsilon 0$, there is an $\eta 0$ such that $|F(A)| < \varepsilon$ for each figure $A \subset [-1/\varepsilon, 1/\varepsilon]$ with $\|A\| < 1/\varepsilon$ and $|A| < \eta$. This observation motivates the next definition.

**Definition 1** *We say that an additive function $F$ of figures in $\mathbb{R}^m$ is continuous whenever, given $\varepsilon 0$, there is an $\eta 0$ such that $|F(A)| < \varepsilon$ for each figure $A \subset [-1/\varepsilon, 1/\varepsilon]^m$ with $\|A\| < 1/\varepsilon$ and $|A| < \eta$.*

It is instructive to describe the topology $\tau$ on the family $\mathcal{F}$ of all figures in $\mathbb{R}^m$ such that each additive function on $\mathcal{F}$ is $\tau$-continuous if and only if it is continuous according to the above definition. To this end, define a metric $\rho$ on $\mathcal{F}$ by letting

$$\rho(A,B) = |(A-B) \cup (B-A)|$$

for each $A, B \in \mathcal{F}$, and for $n = 1, 2, \ldots$, set

$$\mathcal{F}_n = \{A \in \mathcal{F} : A \subset [-n,n]^m \text{ and } \|A\| \leq n\}.$$

Give each $\mathcal{F}_n$ the topology induced by $\rho$, and let $\tau$ be the largest topology on $\mathcal{F}$ for which the imbeddings $\mathcal{F}_n \hookrightarrow \mathcal{F}$ are continuous. The topology $\tau$ is induced by a nonmetrizable uniformity on $\mathcal{F}$, and the completion of the uniform space $(\mathcal{F}, \tau)$ is the space whose points are the equivalence classes of bounded BV sets modulo the sets of measure zero.

The following proposition gives a useful characterization of continuity for additive functions in a figure.

**Proposition 2** (MORTENSEN, PFEFFER). *An additive function $F$ of subfigures of a figure $A$ is continuous if and only if given $\varepsilon 0$, there is a $\theta 0$ such that*

$$|F(B)| < \theta|B| + \varepsilon(\|B\| + 1)$$

*for each figure $B \subset A$.*

**Remark 3** *An additve function $F$ of subfigures of a figure $A$ is $\rho$-continuous if and only if it is absolutely continuous, or alternatively, if and only if given $\varepsilon 0$, there is an $\theta 0$ such that $|F(B)| < \theta|B| + \varepsilon$ for each figure $B \subset A$.*

For $k = 0, \ldots, m$, denote by $\mathcal{H}^k$ the $k$-dimensional Hausdorff measure in $\mathbb{R}^m$. A set $T \subset \mathbb{R}^m$ is called *thin* if its $\mathcal{H}^{m-1}$ measure is $\sigma$-finite. A *gage* on a figure $A$ is a nonnegative function $\delta$ on $A$ such that the set $\{x \in A : \delta(x) = 0\}$ is thin.

A *partition* in a figure $A$ is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are nonoverlapping subfigures of $A$ and $x_i \in A_i$ for $i = 1, \ldots, p$. If $\varepsilon 0$ and $\delta$ is a gage on $A$, we say that $P$ is, respectively, *$\varepsilon$-regular* or *$\delta$-fine* whenever $r(A_i)\varepsilon$ for $i = 1, \ldots, p$ or $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$.

**Definition 4** *A function $f$ on a figure $A$ is called integrable in $A$ if there is an additive continuous function $F$ of subfigures of $A$ satisfying the following condition: given $\varepsilon 0$, there is a gage $\delta$ on $A$ such that*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \varepsilon$$

*for each $\varepsilon$-regular $\delta$-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in $A$.*

It follows from a result of E. J. HOWARD ([2]) that for each figure $B \subset A$, the value $F(B)$ is determined uniquely by the function $f$ restricted to $B$; we call it the *integral* of $f$ over $B$, denoted by $\int_B f\, d\mathcal{H}^m$. The family of all integrable functions in $A$ is denoted by $\mathcal{R}(A)$. Using routine arguments, it is easy to show that $\mathcal{R}(A)$ is a linear space, and that the map $f \mapsto \int_A f\, d\mathcal{H}^m$ is a linear functional on $\mathcal{R}(A)$, which extends the Lebesgue integral.

We say that a sequence $\{B_k\}$ of subfigures of a figure $A$ *converges* to $A$ whenever $\sup \|B_k\| < +\infty$ and $\lim |A - B_k| = 0$. The next theorem shows that the integral has *Hake's property* with respect to this convergence (cf. [5, Chapter VIII, Lemma (3.1)]).

**Theorem 5** *A function $f$ on a figure $A$ is integrable in $A$ if and only if it is integrable in each figure $B \subset A^\circ$ and a finite limit*

$$\lim \int_{B_k} f\, d\mathcal{H}^m = I$$

*exists for each sequence $\{B_k\}$ of subfigures of $A^\circ$ converging to $A$. In either case we have $\int_A f\, d\mathcal{H}^m = I$.*

**Theorem 6** (MORTENSEN, PFEFFER). *Let $f$ be an integrable function in a figure $A$. Then $fg \in \mathcal{R}(A)$ for each Lipschitz function $g$ on $A$. Moreover, if $\{g_k\}$ is an equilipschitz sequence of functions on $A$ that converges pointwise to a Lipschitz function $g$, then*

$$\lim \int_A f g_k\, d\mathcal{H}^m = \int_A fg\, d\mathcal{H}^m .$$

It is unclear, even for $m = 1$, whether the previous theorem still holds when $g$ is a BV function on $A$, which is not Lipschitz.

**Corollary 7** *Let $f$ be an integrable function in a figure $A$, and let $\Lambda(\varphi) = \int_A f\varphi\, d\mathcal{H}^m$ for each $C^\infty$ function $\varphi$ on $\mathbb{R}^m$ vanishing outside a compact set. Then $\Lambda$ is a distribution in $\mathbb{R}^m$ of order at most one whose support is contained in $A$.*

A vector field $v$ on a figure $A$ is called *almost differentiable* at $x \in A^\circ$ whenever

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty .$$

By Stepanoff's theorem ([1, Theorem 3.1.9]), if $v$ is almost differentiable at each point of a set $E \subset A^\circ$, then $v$ is differentiable at $\mathcal{H}^m$-almost all points of $E$.

**Theorem 8** *Let $v$ be a continuous vector field on a figure $A$, and let $T \subset A$ be a thin set. If $v$ is almost differentiable at each $x \in A^\circ - T$, then $div\, v$ is integrable in $A$ and*

$$\int_A div\, v\, d\mathcal{H}^m = \int_{\partial A} v \cdot n\, d\mathcal{H}^{m-1},$$

*where $n$ is the unit exterior normal of $A$ defined $\mathcal{H}^{m-1}$-almost everywhere on $\partial A$.*

Let $A$ and $B$ be figures. A bijective Lipschitz map $\Phi : A \to B$ is called a *lipeomorphism* if the inverse map $\Phi^{-1} : B \to A$ is also Lipschitz. By Rademacher's theorem ([1, Theorem 3.1.6]), each Lipschitz map $\Phi : A \to B$ is differentiable $\mathcal{H}^m$-almost everywhere in $A^\circ$, and we denote by $det\, \Phi$ the determinant of its differential.

**Theorem 9** (NOVIKOV, PFEFFER). *Let $\Phi : A \to B$ be a lipeomorphism from a figure $A$ onto a figure $B$. If $f \in \mathcal{R}(B)$, then $(f \circ \Phi)|det\, \Phi|$ belongs to $\mathcal{R}(A)$ and*

$$\int_A (f \circ \Phi)|det\, \Phi|\, d\mathcal{H}^m = \int_B f\, d\mathcal{H}^m.$$

It is noteworthy that the integral defined by means of coordinate bound figures turns out to be coordinate free. This is not so when figures are replaced by cells in the definition of partitions: Z. BUCZOLICH has shown that in this case the resulting integral is not invariant with respect to rotations.

**Remark 10** *For $m = 1$, the properties stated in Theorems 5 and 8 are usually viewed as characteristic of the Denjoy-Perron integral. Nonetheless, it is not difficult to construct a Denjoy-Perron integrable function on $[0,1]$ that does not belong to $\mathcal{R}([0,1])$.*

## References

[1] H. Federer. *Geometric Measure Theory.* Springer-Verlag, New York, 1969.

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[4] W. F. Pfeffer. *The Riemann Approach to Integration.* Cambridge Univ. Press, New York, 1993.

[5] S. Saks. *Theory of the Integral.* Dover, New York, 1964.