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NEW INTEGRABILITY CLASSES

The problem of integrability amounts to the question when is a trigonometric series $\sum_{k\in Z} c_k e^{ikx}$ a Fourier–Lebesgue series, i.e. when does $(c_k) \in \widehat{L^1}$? Since there are no suitable characterizations of the space $\widehat{L^1}$ we search for subsets of $\widehat{L^1}$ that can be characterized in terms of sequences alone, called the *integrability classes*. In some sense the even integrability classes, i.e. the classes of $(c_k) \in \widehat{L^1}$ satisfying $c_k = c_{-k}$ for all k, are most essential.

The class of all even convex null sequences $\mathcal{K}$ is an integrability class, i.e. $\mathcal{K} \subset \widehat{L^1}$. This class obtained by Young (1913) was extended by Kolmogorov (1923) to the space of quasiconvex null sequences q_0. Since q_0 is the linear span of $\mathcal{K}$, the conclusion that $q_0 \subset \widehat{L^1}$ also follows from Young's theorem. We are referring here to standard BK spaces of two–way even sequences: $\mathbf{c}_0$– the space of null sequences with norm $\| c \|_{\ell^\infty} = \sup_k | c_k |$; ℓ^p, $1 \leq p < \infty$, – the space of all (c_k) such that $\| c \|_{\ell^p} = \left(\sum | c_k |^p\right)^{1/p} < \infty$; bv_0– the space of null sequences of bounded variation under norm $\| c \|_{bv_o} = \| c \|_{\ell^\infty} + \| \Delta c \|_{\ell^1}$; q_0– the space of quasiconvex null sequences with norm $\| c \|_q = \sum k \, | \Delta^2 c_k | + \| c \|_{\ell^\infty} < \infty$.

An enlargement of q_0, introduced by Sidon (1939) and redescribed by Telyakovskii (1973), is the class $\mathcal{ST}$ of all $(c_k) \in \mathbf{c}_0$ such that for some $A_k \downarrow 0$ with $\sum A_k < \infty$, $| \Delta c_k | \leq A_k$. This class was extended by Fomin (1978) to a family of even integrability classes $\mathcal{F}_p$, $p > 1$. A sequence $c = (c_k) \in \mathcal{F}_p$ if and only if $c \in \mathbf{c}_0$ and (1) $\sum_{j=0}^{\infty} 2^{j/q} \| d^j \Delta c \|_{\ell^p} < \infty$. Here $1/p + 1/q = 1$ and $d^j c$ is the jth dyadic section of a sequence c, i.e. $d^j c = \sum_{k=2^j}^{2^{j+1}-1} c_k e^k$. That $\mathcal{F}_p$, $p > 1$ are integrability classes for even series was rediscovered by other authors. It turns out that $\mathcal{F}_\infty = \mathcal{ST}$ and the following chain of proper inclusions is valid for $r > p > 1$: (2) $\mathcal{K} \subset q_0 \subset \mathcal{ST} = \mathcal{F}_\infty \subset \mathcal{F}_r \subset \mathcal{F}_p \subset bv_0 \cap \widehat{L^1}$.

There were several attempts to derive integrability classes larger than Fomin's. Some were shown to reduce to the classes $\mathcal{F}_p$. The proper enlargements are the classes $\mathcal{F}_p^*$, $p > 1$ [4] and their later extensions hv^p, $p > 1$ [1] and HV^p, $p > 1$ [3]. A sequence $c = (c_k) \in hv^p$ if and only if $c \in \mathbf{c}_0$ and there exist sequences of natural numbers (ν_j) nondecreasing and, (k_j) increasing, such that $\nu_j \leq k_{j+1}$ and (3) $\sum_{j=0}^{\infty} \log \frac{k_{j+1}}{\nu_j} \| h^j \Delta c \|_{\ell^1} + \nu_j{}^{1/q} \| h^j \Delta c \|_{\ell^p} < \infty$. Here $h^j c = \sum_{k=k_j}^{k_{j+1}-1} c_k e^k$. The restrictions $\nu_j = k_j$ and (k_j) lacunary in (3) describe the classes $\mathcal{F}_p^*$ and further restrictions to $k_j = 2^j$ give Fomin's classes described by (1). The classes HV^p are certain BK spaces containing hv^p. All of these classes decrease with $p \uparrow \infty$ and satisfy the following proper inclusions for $p > 1$: (4) $\mathcal{F}_p \subset \mathcal{F}_p^* \subset hv^p \subset HV^p \subset bv_0 \cap \widehat{L^1}$.

The corresponding general integrability classes were also determined [5]. By relaxing the ℓ^1–norm in condition (3) new integrability classes were introduced in [5], denoted dl^p and hl^p, $p > 1$, incomparable with hv^p, as well as their BK enlargements $HL^p; p > 1$. These contain sequences outside of bv_0 and give rise to a second chain of proper inclusions valid for all $p > 1$, namely (5) $\mathcal{F}_p \subset dl^p \subset hl^p \subset HL^p \subset \widehat{L^1}$.

A troublesome fact about the above two types of integrability classes was that neither of them contain a special integrability class $\mathcal{W}$, described by a necessary and sufficient condition for integrability of even series with coefficients constant on lacunary blocks, due to M. Weiss (1967). This fact inspired new enlargements of the classes $\mathcal{F}_p$, namely the classes dv^2 and cv^2 [2]. A sequence c belongs to the class dv^2, $\left[\text{respectively to the class, } cv^2\right]$, if and only if $(c_k) \in \mathbf{c}_0$ and satisfies $\sum_{j=0}^{\infty} \left(\sum_{r=1}^{\infty} \| d^{rj} \Delta c \|_{\ell^1}^2 \right)^{1/2} < \infty$, $\left[\text{respectively, } \sum_{j=0}^{\infty} \max_{0 \leq \nu < 2^j} \left(\sum_{r=1}^{\infty} | c_{r2^j+\nu} - c_{(r+1)2^j} |^2 \right)^{1/2} < \infty \right]$, where $d^{rj} c = \sum_{k=r2^j}^{(r+1)2^j-1} c_k e^k$. They are BK spaces and natural extensions of both Fomin's classes $\mathcal{F}_p$, $p > 1$ and the special class $\mathcal{W}$. Furthermore, $dv^2 \subset bv_0$, while cv^2 is incomparable with bv_0 and (6) $\mathcal{W} \cup \mathcal{F}_p \subset dv^2 \subset cv^2 \subset \widehat{L^1}$.

A new and surprising result is that cv^2 is the largest among all these integrability classes, that is a class connecting the chains (3), (5) and (6). Precisely the following proper inclusions are valid for all $p > 1$: $\mathcal{W} \cup \mathcal{F}_p \subset \mathcal{W} \cup hv^p \subset \mathcal{W} \cup HV^p \subset dv^2 \subset cv^2 \cap bv_0 \subset \widehat{L^1}$ and $\mathcal{F}_p \subset dl^p \subset hl^p \subset HL^p \subset cv^2 \subset \widehat{L^1}$.

We conclude with a conjecture about yet a larger family of integrability classes that can be obtained by refining dyadic divisions of dv^2 and cv^2 via a recent extension of the Housdorff–Young theorem.

References

[1] M. Buntinas and N. Tanović–Miller, *New integrability and L^1 convergence*

classes for even trigonometric series, Rad. Mat. **6** (1990), 149–170.

[2] M. Buntinas and N. Tanović–Miller, *New integrability and L^1 convergence classes for even trigonometric series II*, Colloquia Math. Soc. János Bolyai **58**, Approximation Theory, Kecskemét (Hungary) 1990, 103–125

[3] M. Buntinas and N. Tanović–Miller, *Integrability classes and summability*, Israel Math. Conf. Proc. **4** (1991), 75–88.

[4] N. Tanović–Miller, *On integrability and L^1 convergence of cosine series*, Boll. Un. Mat. Ital. (7) **4-B** (1990), 499–516.

[5] N. Tanović–Miller, *New integrability and L^1 convergence classes for general trigonometric series*, Rad. Mat. **8** (1992), to appear.