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LIPSCHITZ MAPS AND ω-LIMIT SETS

Let $\mathcal{C}$ denote the class of continuous self-maps of an interval I. For $f \in \mathcal{C}$ and for each natural number n let $f^1 = f$, $f^{n+1} = f \circ f^n$.

Definition 1 *A set $E \subset I$ is an ω-limit set for $f \in \mathcal{C}$ if there exits $x \in I$ such that the cluster set of the sequence $\{f^n(x)\}$ equals E.*

Necessary and sufficient conditions for a set E to be an ω-limit set for some $f \in \mathcal{C}$ have been obtained in [1] and [2]. In particular, every Cantor set is such an ω-limit set.

It is natural to ask whether the same is true under smoothness restrictions on f. Since any $f \in \mathcal{C}$ maps any of its ω-limit sets onto itself, we study the question in the context of continuous maps of a Cantor set onto itself.

Let $\mathcal{K}$ denote the class of nonempty closed subsets of I. We furnish $\mathcal{K}$ with the Hausdorff metric, producing a compact metric space in which the Cantor sets form a dense G_δ.

Theorem 1 *There exists a residual subset $\mathcal{E}$ of $\mathcal{K}$ such that if $E \in \mathcal{E}$, and $f : E \to E$ is continuous and not the identity on any portion of E, then the following statements are valid:*

1. *There exists a Cantor set $K \subset E$ such that f is not Lipschitz on any portion of K.*
2. *$f(K) = E$.*
3. *If $P = (a, b) \cap E$ is a portion of E contiguous to K, then $f(P)$ is nowhere dense in E.*
4. *If f is Lipschitz on a subset C of E, then $f(C)$ is nowhere dense in E.*
5. *f is nondifferentiable on a dense G_δ on K.*
6. *f maps the set of points of differentiability of E onto a first category subset of E.*

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Observe that if E is an ω-limit set for some $f \in \mathcal{C}$, then unless E consists of only one point, f is not the identity on any portion of E. Thus most closed sets E are not ω-limit sets for Lipschitz functions or for differentiable functions.

The class of ω-limit sets of continuous functions f of zero topological entropy has been characterized in [3]. In particular, each Cantor set E is such an ω-limit set. In this setting, f maps portions of E onto other portions of E. It follows from the theorem that f is not Lipschitz on any portion of E, and that the set of points of differentiability of f is first category in E.

References

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