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T. Natkaniec*, Instytut Matematyki WSP, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland.

Almost Continuity

1 Preliminaries.

1.1 Notations.

Let us establish some of terminology to be used in whole paper. Symbols X, Y will denote topological spaces. $\Re$ denotes the real line (or the one dimensional Euclidean space), I denotes the unit interval $[0,1]$. The symbols N and Q denote the sets of all positive integers and all rationals, respectively.

We use standard topological denotations (see e.g. [19]). If A is a subset of a topological space X then $int(A)$ (or $int_X(A)$) and $fr(A)$ (or $fr_X(A)$) denote the interior of A and the boundary of A, respectively. The closure of A is denoted by $cl(A)$, $cl_X(A)$ or $\overline{A}$. If X is a metric space, $x \in X$ and $\varepsilon > 0$, then $B_X(x,\varepsilon)$ (or simply, $B(x,\varepsilon)$) denotes the open ball centered at x and with the radius ε.

For a subset A of $X \times Y$ we denote by $dom(A)$ and $rng(A)$ the x-projection and y-projection of A; $dom(A) = \{x : \exists y \in Y\ (x,y) \in A\}$, $rng(A) = \{y : \exists x \in X\ (x,y) \in A\}$. If B is a subset of X then $A|B$ denotes the set $A \cap (B \times Y)$. Moreover, if $x \in X$ and $y \in Y$ are fixed, then A_x and A^y denote sections of A; $A_x = \{t \in Y : (x,t) \in A\}$, $A^y = \{t \in X : (t,y) \in A\}$.

We consider a function $f : X \longrightarrow Y$ and its graph (i.e. a subset of $X \times Y$) to be coincident. Symbols $\mathcal{C}onst(X,Y)$, $\mathcal{C}(X,Y)$ and Y^X denote the families of all constant functions, all continuous functions and all functions from X into Y, respectively. We will write $\mathcal{C}onst$ and $\mathcal{C}$ instead of $\mathcal{C}onst(X,Y)$ and $\mathcal{C}(X,Y)$ when X and Y are fixed. Symbol $C(f)$ denotes the set of all continuity points of f. If we consider a function f defined on $\Re$ then the symbols $C^-(f,x)$ and $C^+(f,x)$ denote the left and the right cluster sets of f

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at the point x. If f is a real function defined on X then the notation $[f > 0]$ means the set $\{x \in X : f(x) > 0\}$. Likewise for $[f = 0]$, $[f \neq 0]$, etc.

If A is a set then by $card(A)$ we shall denote the cardinality of A. Cardinals will be identified with initial ordinals.

We shall use the following set theoretical assumptions:

$A(c)$: the union of less than 2^ω many first category subsets of $\Re$ is again of the first category.

$A(m)$: the union of less than 2^ω many subsets of measure zero of $\Re$ is again of measure zero.

It is well-known that these conditions follow from Martin's Axiom and therefore also from the Continuum Hypothesis (see e.g. [56]). If not explicitly stated otherwise, we shall work in (ZFC) without further assumptions.

1.2 Basic definitions.

A function $f : X \longrightarrow Y$ is *almost continuous in the sense of Stallings* iff for any open set $U \subset X \times Y$ containing f, U contains a continuous function $g : X \longrightarrow Y$ [60]. The class of all almost continuous functions from X into Y is denoted by $\mathcal{A}(X,Y)$, or $\mathcal{A}$ when X and Y are fixed.

Clearly any continuous function is almost continuous. There are, however, many almost continuous real functions which are not continuous. The following two examples of non-continuous, almost continuous functions are "classical".

Example 1.1 *Let $f_0 : [-1,1] \longrightarrow [-1,1]$ be defined by $f_0(x) = sin(1/x)$ for $x \neq 0$ and $f_0(0) = 0$. It is easy to observe that f_0 is almost continuous.*

Example 1.2 *Let $f : I \longrightarrow I$ be defined by $f(x) = \overline{lim}_{n\longrightarrow\infty}(a_1 + \ldots a_n)/n$, where a_i are given by the unique nonterminating binary expansion of the number $x = (0.a_1a_2\ldots)$. Then f is almost continuous [6].*

Note that the last function is dense in I^2. Other examples of almost continuous, dense in I^2 functions are constructed in [40], [21], [3]. One can construct such examples using the following notion of blocking sets. This notion was introduced by Kellum and Garret [40], and was used later in many papers, e.g. in [33], [34], [38], [39], [26] and [47], [48], [49] and [50].

Observe that if a function $f : X \longrightarrow Y$ is not almost continuous then there exists a closed set $F \subset X \times Y$ such that $F \cap f = \emptyset$ and $F \cap g \neq \emptyset$ for each continuous function $g : X \longrightarrow Y$. Every such set is called a blocking set for f in $X \times Y$. If no proper subset of F is a blocking set of f in $X \times Y$, F is said to be a minimal blocking set for f in $X \times Y$. If set F is a (minimal) blocking set of some function $f : X \longrightarrow Y$, then F is said to be a (minimal) blocking set in $X \times Y$.

Remark 1.1 *A function $f : X \longrightarrow Y$ is almost continuous iff it intersects every blocking set in $X \times Y$.*

We say that a topological space X *has the fixed point property* iff for any continuous function f from X into X there exists a point $x \in X$ such that $f(x) = x$. Stallings introduced the notion of almost continuity in order to prove a generalization of the Brower fixed point theorem. Note that for a non-degenerate Hausdorff space X with the fixed point property the diagonal $\{(x,x) : x \in X\}$ is a blocking set in $X \times X$. Therefore we obtain the following property of almost continuous functions.

Theorem 1.1 *If X is a Hausdorff space with the fixed point property then each almost continuous function $f : X \longrightarrow X$ has a fixed point [60].*

Theorem 1.2 *Suppose that X is a compact space and $f : X \longrightarrow Y$ is not almost continuous. Then*

(1) there exists a minimal blocking set K of f in $X \times Y$, and

(2) $dom(K)$ is contained in a component of X,

(3) if one of the following conditions holds:

(i) X is perfectly normal and Y is an interval in $\Re^k$, ($k \in N$),

(ii) X is an interval in $\Re$ and Y is a convex subspace of $\Re^k$, ($k \in N$),

(iii) Y is an ε-absolute retract (see [37] for definitions),

then $dom(K)$ is a non-degenerate connected set,

(4) $rng(K) = Y$.

P r o o f . (1) is proved in [36].
(2) Suppose that K is a blocking set for f in $X \times Y$ and S_1, S_2 are different components of X with $dom(K) \cap S_1 \neq \emptyset \neq dom(K) \cap S_2$. Since X is compact, there exists a clopen set $A_1 \subset X$ such that $S_1 \subset A_1$ and $S_2 \subset A_2 = X \setminus A_1$ (see e.g. [19], Theorem 8, p. 431). Since $K|A_i$ $(i = 1, 2)$ are not blocking for f, there exist continuous functions $g_i : X \longrightarrow Y$ such that $g_i \cap (K|A_i) = \emptyset$. Thus $(g_1|A_1) \cup (g_2|A_2)$ is continuous and disjoint from K, a contradiction.

(3.*i*) Suppose that $dom(K)$ is not connected. Let (A_1, A_2) be a partition of $dom(K)$ into disjoint, non-empty sets which are clopen in $dom(K)$. Let $g_1, g_2 : X \longrightarrow Y$ be continuous and such that $g_i \cap (K|A_i) = \emptyset$ for $i = 1, 2$. Let $C = fr_X(A_1)$. Since X is perfectly normal, there exists a decreasing sequence of open sets $(U_n)_n$ such that $C = \bigcap_{n=1}^{\infty} U_n$. Since g_1, g_2 are bounded, there exists a cube $J_0 \subset Y$ such that $rng(g_i) \subset J_0$ for $i = 1, 2$. For each $n \in N$ let $h_n : X \longrightarrow J_0$ be a continuous extension of the function $(g_1|(\overline{A_1} \setminus U_n)) \cup (g_2|(\overline{A_2} \setminus U_n))$. Since K is blocking, there exists $x_n \in U_n \cap dom(K)$ such that $(x_n, h_n(x_n)) \in K$. Let (x, y) be a limit point of the sequence $(x_n, h_n(x_n))_n$. Then $x \in C \cap dom(K) = fr_{dom(K)}(A_1)$, which is impossible.

Proofs of the statements (3.*ii*) and (4) are the same as in [34] (when $X = Y = \Re$). (3.*iii*) is proved in [37], Theorem 5.2.

Q.E.D.

Corollary 1.1 *If $f : \Re \longrightarrow \Re^k$ is not almost continuous then there exists a blocking set $K \subset \Re \times \Re^k$ for f such that $dom(K)$ is a non-degenerate interval (cf. [34]).*

Now we try to shed some light on the problem suggested in Remark 3 of [36].

Theorem 1.3 *(on homogeneity of minimal blocking sets.) Assume that $K \subset I \times \Re^k$ is a minimal blocking set, $U_1 = (a_1, a_2) \subset I$, U_2 is an open interval in $\Re^k$ and $U_1 \times U_2 \cap K \neq \emptyset$. Then:*

(1) $int(dom(K \cap (U_1 \times U_2))) \neq \emptyset$ or K intersects every $f \in \mathcal{C}(U_1, U_2)$,

(2) $dom(K \cap (U_1 \times U_2))$ is dense in itself or $\overline{U_2} \subset K_x$ for some $x \in U_1$.

P r o o f . (1) Suppose that $f : U_1 \longrightarrow U_2$ is continuous and $f \cap K = \emptyset$. It follows from minimality of K that $h \cap (K \setminus (U_1 \times U_2)) = \emptyset$ for some continuous function $h : I \longrightarrow \Re^k$. Since K is blocking, $h \cap K \cap (U_1 \times U_2) \neq \emptyset$. Since

$h \cap K$ is compact and $dom(h \cap K) \subset U_1$, we can choose reals b_1, b_2 such that $a_1 < b_1 < m_1 = min(dom(h \cap K)) \leq m_2 = max(dom(h \cap K)) < b_2 < a_2$. Since $A = rng(f|[b_1, b_2]) \cup rng(h \cap K)$ is a compact subset of U_2, there exists a closed interval $J \subset U_2$ such that $A \subset int(J)$. Note that $h \cap K \subset (b_1, b_2) \times int(J)$.

Suppose that $int(dom(K \cap (U_1 \times U_2))) = \emptyset$. Since $K_0 = K \cap ([b_1, b_2] \times J)$ is compact, $dom(K_0)$ is nowhere dense and we can choose intervals $[t_1, t_2] \subset (b_1, m_1) \setminus dom(K_0)$ and $[v_1, v_2] \subset (m_2, a_2) \setminus dom(K_0)$ such that $rng(h|[t_1, t_2]) \cup rng(h|[v_1, v_2]) \subset J$. Then $a_1 < t_1 < t_2 < m_1 \leq m_2 < v_1 < v_2 < a_2$. Let g_1, g_2 be segments in $I \times J$ with end-points $(t_1, h(t_1))$, $(t_2, f(t_2))$ and $(v_1, f(v_1))$, $(v_2, h(v_2))$, respectively. Then the function $g = h|(I \setminus (t_1, v_2)) \cup g_1 \cup g_2 \cup f|(t_2, v_1)$ is continuous and disjoint with K, a contradiction.

(2) Suppose that x is isolated in $dom(K \cap (U_1 \times U_2))$. Let $V \subset U_1$ be an interval such that $\{x\} = V \cap dom(K \cap (U_1 \times U_2))$. Then, by (1), $rng(K \cap (V \times U_2)) = U_2$ and therefore $\overline{U_2} \subset K_x$.

Q.E.D.

Theorem 1.4 *Let $f : \Re \longrightarrow \Re$ be a function such that $f \cap cl(u) \neq \emptyset$ for any upper semi-continuous function u, defined on non-degenerate interval. Then f is almost continuous [38].*

A pair of topological spaces X, Y will be called a (K, G) pair (Kellum-Garret pair) iff there exists a family $\mathcal{F}$ of blocking sets in $X \times Y$ such that

(1) if $f \notin \mathcal{A}(X, Y)$ then in $\mathcal{F}$ there exists a blocking set for f,

(2) $card(dom(F)) \geq card(\mathcal{F})$ for any $F \in \mathcal{F}$.

A family which satisfies the conditions (1) and (2) will be called a blocking family for the pair (X, Y).

Proposition 1.1 *The following pairs (X, Y) are of (K, G) type.*

(1) X is compact, perfectly normal and Y is a non-degenerate interval in $\Re^k$,

(2) X is a compact interval in $\Re$ and Y is a convex subspace of $\Re^k$,

(3) X is an interval in $\Re$ and Y is a convex subspace of $\Re^k$.

P r o o f . In the cases (1) and (2) we can take the families of minimal blocking sets in $X \times Y$ as the blocking families for (X, Y) (cf. Theorem 1.2). In the case (3), X can be decomposed into a countable sequence $(I_n)_n$ of closed intervals such that $I_n \cap I_{n+1} \neq \emptyset$ for $n \in N$. One can prove that if $f \notin \mathcal{A}(X, Y)$ then $f|I_n \notin \mathcal{A}(I_n, Y)$ for some $n \in N$ (see Lemma 2.3 below). Let $\mathcal{F}_n$ be a blocking family for (I_n, Y). Then the union of all $\mathcal{F}_n$, $n \in N$, is a blocking family for the pair (X, Y).

Q.E.D.

Proposition 1.2 *$(\Re^k, \Re^m)$ is a (K, G) pair for all $k, m \in N$.*

P r o o f . Obviously $card(\mathcal{K}) \leq 2^\omega$ for every blocking family $\mathcal{K}$ in $\Re^k \times \Re^m$ (in fact it is easy to see that $card(\mathcal{K}) = 2^\omega$). Thus it is sufficient to prove that $card(dom(K)) = 2^\omega$ for every blocking set K in $\Re^k \times \Re^m$. Suppose that $card(dom(K)) < 2^\omega$. Then there exists an increasing sequence $(r_n)_n$ of positive reals such that $lim_{n \longrightarrow \infty} r_n = \infty$ and $S_n \cap dom(K) = \emptyset$, where S_n denotes the $(k-1)$-dimensional sphere in $\Re^k$ centered at 0 and with radius r_n. Fix $n \in N$ and put $A_n = \overline{B(0, r_n)} \setminus B(0, r_{n-1})$. Then for each $i \in N$, $K_{n,i} = dom(K \cap (A_n \cap [-i, i]^m))$ is compact and $card(K_{n,i}) < 2^\omega$. Hence $K \cap (A_n \cap [-i, i]^m)$ is not blocking in $A_n \cap [-i, i]^m$, so either there exists a continuous function $f : A_n \longrightarrow [-i, i]^m$ such that $f \cap K = \emptyset$ or $\{x\} \times [-i, i]^m \subset K$ for some $x \in A_n$. Note that there exist i_n and a continuous function $f_n : A_n \longrightarrow [-i_n, i_n]^m$ such that $f_n \cap K = \emptyset$. Indeed, suppose that for each $i \in N$ there exists $x_i \in A_n$ such that $\{x_i\} \times [-i, i]^m \subset K$. Let x_0 be a limit point of the sequence $(x_i)_i$. Since K is closed $\{x_0\} \times \Re^m \subset K$, which contradicts the assumption that K is blocking.

Since K_{n,i_n} is compact, $dist\,(K_{n,i_n}, S_{n-1} \cup S_n) > 0$, so we can assume that $f_n|(S_{n-1} \cup S_n) \equiv 0$. Then $f = \bigcup_{n=1}^{\infty} f_n$ is continuous and disjoint with K, a contradiction.

Q.E.D.

1.3 Collation with other classes of functions.

1.3.1 Almost continuity and continuity.

T. Husain [28] has introduced another notion of almost continuity. A function $f : X \longrightarrow Y$ is *almost continuous in the sense of Husain* (H-almost continuous) iff for each $x \in X$, if $V \subset Y$ is a neighbourhood of $f(x)$ then

$f^{-1}(V)$ is dense in a some neighbourhood of x. Relationships between continuity, almost continuity (in the sense of Stallings) and H-almost continuity are studied in [20], [44], [58], [59]. A function $f : X \longrightarrow Y$ is *of Cesaro type* iff there exist non-empty open sets $U \subset X$ and $V \subset Y$, such that $f^{-1}(y)$ is dense in U for each $y \in V$ (cf. [59]). The class of all functions of Cesaro type for which $U = X$ and $V = Y$ will be denoted by $\mathcal{D}^*(X,Y)$ (or $\mathcal{D}^*$ when X and Y are fixed). Now let (Y,ρ) be a metric space. A function $f : X \longrightarrow Y$ is called *cliquish* iff for each $\varepsilon > 0$, every non-empty open set $U \subset X$ contains a non-empty open set V such that $\rho(f(x), f(y)) < \varepsilon$ whenever $x, y \in V$.

Theorem 1.5 *Let X be a regular locally connected Baire space. Then for every real function f, f is continuous iff f is almost continuous, H-almost continuous, and not of Cesaro type [58] (and [60] for $X = \Re$).*

Example 1.3 *There exists an almost continuous and H-almost continuous function $f : I \longrightarrow \Re^2$ which is not of Cesaro type and not continuous.*

Indeed, let $f_1 = id_I$, $f_2 : I \longrightarrow \Re$ be almost continuous, $f_2 \in \mathcal{D}^*$ and let $f = (f_1, f_2)$. Then f is almost continuous (see Theorem 4.4 below), H-almost continuous injection (so it is not of Cesaro type) and it is not continuous.

Theorem 1.6 *Let X be a regular locally connected Baire space. If a real function defined on X is almost continuous and not of the Cesaro type then it is cliquish [58].*

Note that there exist almost continuous real functions defined on I which are of Cesaro type (see e.g. Example 1.2). Clearly such functions have no points of continuity. Moreover, J. Ceder gave an example (under CH) of an almost continuous function $f : I \longrightarrow \Re$ such that $f|A$ is discontinuous whenever A is uncountable ([15], see also [38]).

1.3.2 Almost continuity, connectivity and other Darboux-like properties.

Theorem 1.7 *If X is a connected T_1 space, Y is a hereditarily normal Hausdorff space and $f : X \longrightarrow Y$ is almost continuous, then $rng(f)$ is connected.*

P r o o f . Suppose that $rng(f)$ is not connected. Since Y is hereditarily normal, there exist disjoint open sets $U, V \subset Y$ such that $rng(f) \subset U \cup V$

and $rng(f) \cap U \neq \emptyset \neq rng(f) \cap V$ (see e.g. [19], Theorem 6, p. 96). Fix $x_1, x_2 \in X$ such that $f(x_1) \in U$ and $f(x_2) \in V$. Then $G = X \times (U \cup V) \setminus ((\{x_1\} \times (Y \setminus U)) \cup (\{x_2\} \times (Y \setminus V)))$ is an open neighbourhood of f and it includes a continuous function $g : X \longrightarrow Y$. Since $g(x_1) \in U$ and $g(x_2) \in V$, $rng(g) \cap U \neq \emptyset \neq rng(g) \cap V$. Hence $rng(g)$ is not connected, which contradicts the continuity of g. Q.E.D.

Theorem 1.8 *If $X \times Y$ is a hereditarily normal Hausdorff space, X is connected and $f : X \longrightarrow Y$ is almost continuous, then f is a connected subset of $X \times Y$ [60].*

Corollary 1.2 *If X is a connected hereditarily normal Hausdorff space and Y is a discrete space then $\mathcal{A}(X,Y) = Const(X,Y)$.*

Example 1.4 *There exists a connected space X and an almost continuous bijection $f : X \longrightarrow X$ such that $f = f^{-1}$ and f is not connected in $X \times X$ (thus f is not continuous).*

Indeed, let X be the unit interval with the topology $\tau = \{U \subset I : 0 \in U\} \cup \{\emptyset\}$ and let $f : X \longrightarrow X$ be the function given by $f(x) = x$ for $x \in (0,1)$ and $f(x) = 1 - x$ for $x \in \{0,1\}$. Then f is almost continuous. In fact, if G is a neighbourhood of f in $X \times X$ then $(x,0) \in G$ for each $x \in I$ and consequently, G includes a constant function $g \equiv 0$. Since $\{(0,1)\}$ is clopen in f, f is not connected.

Theorem 1.9 *Assume that $f : X \longrightarrow Y$. Then*

(1) if Y_0 is a subspace of Y, $rng(f) \subset Y_0$ and $f \in \mathcal{A}(X,Y_0)$, then $f \in \mathcal{A}(X,Y)$,

(2) for any function $f : X \longrightarrow Y$ there exists an extension Y' of Y for which $f \in \mathcal{A}(X,Y')$ [36].

(3) if J is an interval in $\Re$, $f \in \mathcal{A}(X,\Re)$ and $rng(f) \subset J$, then $f \in \mathcal{A}(X,J)$. (Hence $f \in \mathcal{A}(X,\Re)$ iff $f \in \mathcal{A}(X,rng(f))$ for each real-valued function f defined on X.)

Proof. (1) is obvious. To prove (3) assume that $f \in \mathcal{A}(X, \Re)$, $rng(f) \subset J$ and J is an interval in $\Re$, e.g. of the form $(a, b|$. Let $G \subset X \times J$ be an open neighbourhood of f in $X \times J$. Then $G_1 = G \cup (X \times (b, \infty))$ is a neighbourhood of f in $X \times \Re$. Let $g_1 : X \longrightarrow \Re$ be a continuous function contained in $X \times \Re$. Then $g = min(g_1, b)$ is continuous and contained in G. Finally note that $rng(f)$ is an interval (see Theorem 1.7) and therefore $f \in \mathcal{A}(X, \Re)$ iff $f \in \mathcal{A}(X, rng(f))$.

Q.E.D.

Example 1.5 *There exist* Y *and* $f \in \mathcal{A}(I, Y) \setminus \mathcal{A}(I, rng(f))$ *[36].*

Indeed, let Y be the space X defined in Example 1.4 and let $f : I \longrightarrow Y$ be given by $f(x) = x$ for $x \neq 0$ and $f(x) = 1$ for $x = 0$. As in Example 1.4 one can verify that $f \in \mathcal{A}(I, Y)$. Moreover, $rng(f)$ is a discrete space (of cardinality 2^ω), and therefore only constant functions belong to the family $\mathcal{A}(I, rng(f))$.

Note that it follows from the above example that there are almost continuous functions defined on connected spaces whose images are not connected.

Almost continuous retractions of cubes $[-1, 1]^n$ are described in [35], [36].

Now we shall consider the following classes of functions from X into Y:

$\mathcal{D}(X, Y)$ — the family of all Darboux functions. f is a *Darboux function* iff $f(C)$ is connected whenever C is connected in X.

$\mathcal{C}onn(X, Y)$ — the family of all connectivity functions. f is a *connectivity function* iff $f|C$ is a connected subset of $X \times Y$ whenever C is connected in X.

$\mathcal{E}xt(X, Y)$ — the class of all extendable functions. f is *extendable* iff there exists $g \in \mathcal{C}onn(X \times I, Y)$ such that $f(x) = g(x, 0)$ for each $x \in X$.

We shall write $\mathcal{D}$, $\mathcal{C}onn$ and $\mathcal{E}xt$, respectively, when X and Y are fixed. Now let $X = I$, $Y = \Re$ and

$\mathcal{L}$ - the class of Lebesgue measurable functions from I into $\Re$.

$\mathcal{B}$ - the class of Borel measurable functions from I into $\Re$.

$\mathcal{J}_1$ - the class of pointwise limits of sequences of functions from I into $\Re$ which have only discontinuities of the first kind.

$\mathcal{R}_1$ - the class of pointwise limits of sequences of functions from I into $\Re$ which are continuous from the right.

$\mathcal{B}_1$ - the first class of Baire of functions from I into $\Re$.

Note that $\mathcal{B}_1 \subset \mathcal{R}_1 \subset \mathcal{J}_1 \subset \mathcal{B} \subset \mathcal{L}$ [52].

Theorem 1.10 *In the class of all real functions defined on I the following relations hold:*

(1) $\mathcal{E}xt \subset \mathcal{A} \subset \mathcal{C}onn \subset \mathcal{D}$ [60].

(2) $\mathcal{A} \neq \mathcal{C}onn$ (see [9] and [18], [32], [53] and [60] for examples).

(3) $\mathcal{L} \cap \mathcal{E}xt \neq \mathcal{L} \cap \mathcal{A}$ and $\mathcal{B}_1 \cap \mathcal{E}xt = \mathcal{B}_1 \cap \mathcal{A}$ [7].

(4) $\mathcal{B}_1 \cap \mathcal{A} = \mathcal{B}_1 \cap \mathcal{C}onn$ and $\mathcal{R}_1 \cap \mathcal{A} \neq \mathcal{R}_1 \cap \mathcal{C}onn$ [5].

Problem 1.1 *For which $\mathcal{X} \in \{\mathcal{B}, \mathcal{J}_1, \mathcal{R}_1\}$ is it true that $\mathcal{X} \cap \mathcal{E}xt = \mathcal{X} \cap \mathcal{A}$? [7]*

Note that the inclusion $\mathcal{C}onn(X,Y) \subset \mathcal{D}(X,Y)$ holds for each pair of topological spaces X, Y. However this is not true for all inclusions (1) from 1.10, even for real functions defined on cubes.

Theorem 1.11 *If $k > 1$ then $\mathcal{C}onn(I^k, I) \subset \mathcal{A}(I^k, I)$ [60].*

Example 1.6 *There exists $f \in \mathcal{A}(I^2, I) \setminus \mathcal{D}(I^2, I)$.*

Indeed, let A_0 be a closed segment with end-points $(0,1)$ and $(1,1)$, and for each $n \in N$ let A_n be a closed segment with end-points $(1/n, 0)$ and $(1/n, 1)$. Let $A = \bigcup_{n=0}^{\infty} A_n$ and $B = A \cup \{(0,0)\}$. Observe that B is connected and for each non-degenerate continuum $C \in I^2$ either $C \subset A$ or $card(C \setminus B) = 2^\omega$. In fact, let us assume that C is a non-degenerate continuum and $C \setminus A \neq \emptyset$. If $dom(C \setminus A) \cap (0,1| \neq \emptyset$ or $C \subset \{0\} \times I$ then the assertion is obvious. Otherwise, $dom(C)$ is a non-degenerate interval and there exists $\delta > 0$ such that $(0,\delta) \times \{1\} \subset C$. Let $y < 1$ be such that $x = (0,y) \in C$. If $B(x,r) \cap (0,1| \times I = \emptyset$ for some $r > 0$ then $\{0\} \times J \subset C$ for some closed non-degenerate interval J. Otherwise there exist an increasing sequence $(k_n)_n$ and

$z \in (y,1)$ such that and $\{1/k_n\} \times [z,1] \subset C$ for each $n \in N$ and therefore, $\{0\} \times [z,1] \subset C$ and $card(C \setminus B) = 2^\omega$.

Let $(K_\alpha)_{\alpha<2^\omega}$ be a sequence of all minimal blocking sets K in $I^2 \times I$ such that $dom(K) \setminus A \neq \emptyset$. Let $(x_\alpha, y_\alpha)_{\alpha<2^\omega}$ be a sequence of points such that $(x_\alpha, y_\alpha) \in K_\alpha$, $x_\alpha \in dom(K_\alpha) \setminus B$ and $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Define $f : I^2 \longrightarrow I$ by $f(x) = 1$ for $x = (0,0)$, $f(x_\alpha) = y_\alpha$ for $\alpha < 2^\omega$ and $f(x) = 0$ otherwise. Observe that f intersects each blocking set in $I^2 \times I$. In fact, let F be a blocking set and let $K \subset F$ be a minimal blocking set. If $K = K_\alpha$ for some $\alpha < 2^\omega$ then $(x_\alpha, y_\alpha) \in f \cap K$. If $K \neq K_\alpha$ for each $\alpha < 2^\omega$ then $f(x) = 0$ for each $x \in dom(K)$. Since $rng(K) = I$, $(x,0) \in f \cap K$ for some $x \in dom(K)$. Thus $f \in \mathcal{A}(I^2, I)$. Since $f(B) = \{0,1\}$, $f \notin \mathcal{D}(I^2, I)$.

Let $\mathcal{D}_P(X,Y)$ denote the family of all Darboux functions in the sense of Pawlak [51], i.e. functions $f : X \longrightarrow Y$ such that $f(L)$ is connected whenever L is an arc in X.

Theorem 1.12 *If Y is hereditarily normal, then $\mathcal{A}(X,Y) \subset \mathcal{D}_P(X,Y)$.*

Proof. Let L be an arc in X and let $f : X \longrightarrow Y$ be almost continuous. It will be shown in Theorem 2.1, that $f|L \in \mathcal{A}(L,Y)$. By Theorem 1.7, $rng(f|L)$ is connected. Q.E.D.

In connection with the condition (3) of Theorem 1.10 we have the following Lipiński's example.

Example 1.7 *Let $X = [-1,1] \times \Re$ and $Y = [-1,1]$. Then $\mathcal{B}_1(X,Y) \cap (\mathcal{D}(X,Y) \setminus \mathcal{A}(X,Y) \neq \emptyset$ [42].*

Let $f : [-1,1] \times \Re \longrightarrow [-1,1]$ be given by $f(x,y) = f_0(x)$, where f_0 is the function defined in Example 1.1. Then f has required properties [42].

More information about relationships between almost continuity and other Darboux-like classes one can found in Gibson's papers, e.g. [23], [24], [55].

1.4 The local characterization.

Many authors have considered the local property of Darboux (i.e. the intermediate value property) [10] or local connectivity of a real function [22] and the sets of those points at which a real function of a real variable has the local Darboux property [43] or local connectivity property [54]. The local characterization of almost continuity is given in [31] and in that paper one can find proofs of the next three theorems.

We say that a function f from $\Re$ into $\Re$ is *almost continuous* at a point $x \in \Re$ from the right iff

(1) $f(x) \in C^{+}(f,x)$,

(2) there is a positive ε such that for any neighbourhood G of $f|[x,\infty)$, arbitrary $y \in (\underline{lim}_{t\longrightarrow x^+} f(t), \overline{lim}_{t\longrightarrow x^+} f(t))$, arbitrary neighbourhood U of the point (x,y) and arbitrary $t \in (x, x+\varepsilon)$ there exists a continuous function $g : [x, x+\varepsilon] \longrightarrow \Re$ such that $g \subset G \cup U$, $g(x) = y$ and $g(t) = f(t)$.

Analogously we define the notion of almost continuity at a point from the left. If f is almost continuous at a point x from both sides then we say that f is *almost continuous* at x or that x is a point of almost continuity of f.

Theorem 1.13 *A function $f : \Re \longrightarrow \Re$ is almost continuous iff f is almost continuous at every point x of $\Re$.*

For arbitrary function $f : \Re \longrightarrow \Re$ let $A(f)$, $Conn(f)$ and $D(f)$ denote the sets of all points at which f is almost continuous, connectivity and has the Darboux property, respectively.

Theorem 1.14 *For every function $f : \Re \longrightarrow \Re$,*

$$(C(f), A(f), Conn(f), D(f))$$

is an increasing sequence of G_δ-sets.

Theorem 1.15 *For every G_δ-set $A \subset \Re$ there exists a function $f : \Re \longrightarrow \Re$ such that $A(f) = A$.*

Problem 1.2 *Find necessary and sufficient conditions for a sequence (A, B, C, D) of subsets of $\Re$ to exist a function $f : \Re \longrightarrow \Re$ such that $(A,B,C,D) = (C(f), A(f), Conn(f), D(f))$.*

2 Restrictions and extensions.

Theorem 2.1 *If X_0 is a closed subspace of X and $f \in \mathcal{A}(X,Y)$, then $f|X_0 \in \mathcal{A}(X_0,Y)$ [60].*

The following example is a bounded version of Lipiński's function from Example 1.7 and shows that the assumption about X_0 is important.

Example 2.1 *There exists an almost continuous function f from $[-1,1] \times [-1,1]$ into $[-1,1]$ for which the restriction $f|(-1,1) \times (-1,1)$ is not almost continuous.*

Indeed, let $f : [-1,1] \times [-1,1] \longrightarrow [-1,1]$ be defined by $f(x,y) = f_0(x)$, where $f_0 : [-1,1] \longrightarrow [-1,1]$ is the function defined in Example 1.1. It will be proved in Corollary 4.2 that f is almost continuous. We shall verify that $f|A$ is not almost continuous for $A = (-1,1) \times (-1,1)$. Let $h : (-1,1) \longrightarrow \Re$ be an increasing homeomorphism. Put $B_0 = \{(x,y,z) : |x| < e^{-h^2(y)}/10$ and $|z| < e^{-h^2(y)}/10\}$ and $B_1 = \{(x,y,z) : x \neq 0$ and $|z - sin(1/x)| < 1/10\}$. Clearly B_0 and B_1 are open and $f|A \subset B_0 \cup B_1$. Suppose that there exists a continuous function $g : A \longrightarrow [-1,1]$ contained in $B_0 \cup B_1$. Then $(0,0,g(0,0)) \in B_0$ and $|g(0,0)| < 1/10$ and therefore there is a positive δ such that $|g(x,0)| < 1/10$ for $x \in (-\delta,\delta)$. Fix $x_0 \in (0,\delta)$ such that $sin(1/x_0) = 1$ and $y_0 \in (0,1)$ for which $x_0 > e^{-h^2(y_0)}/10$. Then $(x_0, y_0, g(x_0,y_0)) \in B_1$, so $g(x_0,y_0) > 9/10$. Observe that the x_0-section g_{x_0} of g (given by $g_{x_0}(y) = g(x_0,y)$ for $y \in (-1,1)$) is continuous, $g_{x_0}(0) < 1/10$ and $g_{x_0}(y_0) > 9/10$. Moreover, $|g_{x_0}(y)| < 1/10$ for $(x_0,y,g(x_0,y)) \in B_0$ and $|g_{x_0}(y)| > 9/10$ if $(x_0,y,g(x_0,y)) \in B_1$. Since $g_{x_0} \subset (B_0 \cup B_1)_{x_0}$, g_{x_0} does not have the Darboux property, which contradicts the continuity of g_{x_0}.

Lemma 2.1 *If X is a second countable zero-dimensional space then each function defined on X is almost continuous.*

Proof. Fix $f : X \longrightarrow Y$ and an open neighbourhood $G \subset X \times Y$ of f. Then $G = \bigcup_{n=1}^{\infty} U_n \times V_n$, where the sets U_n are clopen in X, the sets V_n are open in Y and U_n, V_n are non-empty. For any $n \in N$ choose $y_n \in V_n$. Then $g = \bigcup_{n=1}^{\infty}(U_n \setminus \bigcup_{i<n} U_i) \times \{y_n\}$ is a continuous function defined on X and contained in G.

Q.E.D.

Corollary 2.1 *Every function defined on a boundary subset of $\Re$ is almost continuous (see [40] for real functions defined on compact subsets of I).*

Lemma 2.2 *Let A be a subset of I and let $f : A \longrightarrow \Re^k$ be a function such that $f|cl_A(J) \in \mathcal{A}(cl_A(J), \Re^k)$ for every component J of $int(A)$. Then f is almost continuous.*

P r o o f . Let $G \subset A \times \Re^k$ be a neighbourhood of f. For every component J of $int(A)$ choose an open interval $U_J \subset I$ such that $cl_A(J) \subset U_J$ and:

(i) U_J is clopen in A,

(ii) if a is a left (right) end-point of J and $a \notin A$ then $inf(U_J) = a$ ($sup(U_J) = a$),

(iii) if a is a left (right) end-point of J and $a \in A$ then there exists a neighbourhood V_a of $f(a)$ such that $(inf(U_J), a) \times V_a \subset G$ ($(a, sup(U_J)) \times V_a \subset G$),

(iv) if J_1, J_2 are components of $int(A)$ then $U_{J_1} \cap U_{J_2} = \emptyset$ or $U_{J_1} \subset U_{J_2}$ or $U_{J_2} \subset U_{J_1}$.

Put $B = A \setminus \bigcup_J U_J$. Then there exist open sets U_i, V_i, $i \in N$ such that

(v) $B \subset \bigcup_{i=1}^{\infty} U_i$,

(vi) $\bigcup_{i=1}^{\infty} U_i \times V_i \subset G$,

(vii) U_i are pairwise disjoint and clopen in A,

$(viii)$ for any component J of $int(A)$ and for each $i \in N$ either $U_J \cap U_i = \emptyset$ or $U_J \subset U_i$.

Fix an arbitrary component J of $int(A)$. Since $f|cl_A(J)$ is almost continuous, there exists a continuous function $g_J : cl_A(J) \longrightarrow \Re^k$ such that $g_J \subset G$ and $g_J|fr_A(J) = f|Fr_A(J)$. Let $g_J^* : U_J \longrightarrow \Re^k$ be an extension of g_J given by $g_J^* = (inf(U_J), inf(J)| \times \{f(inf(J))\} \cup g_J \cup [sup(J), sup(U_J)) \times \{f(sup(J))\}$.

Observe that $A = \bigcup_{i=1}^{\infty} U_i \cup \bigcup_{J \not\subset \bigcup_i U_i} U_J$. For each $n \in N$ choose $y_n \in V_n$. Then $g = \bigcup_{i=1}^{\infty} U_i \times \{y_i\} \cup \bigcup_{J \not\subset \bigcup_i U_i} g_J^*$ is a continuous function defined on A and contained in G.

Q.E.D.

The following lemma is proved in [30] for real functions defined on the real line.

Lemma 2.3 *Let an interval $J \subset \Re$ be a union of countably many of closed intervals I_n such that $int(I_n) \cap int(I_m) = \emptyset$ for $m \neq n$ and $I_n \cap I_{n+1} \neq \emptyset$ for $n \in N$, and let Y be a convex subspace of $\Re^k$. For any function $f : J \longrightarrow Y$ if $f|I_n$ is almost continuous for each n then f is almost continuous, too.*

P r o o f . This proof is analogous to the corresponding proof in [30].

Corollary 2.2 *A function $f : \Re \longrightarrow \Re$ is almost continuous iff $f|[k, k+1]$ is almost continuous for each integer k [34].*

Note that the analogous result does not hold for functions of two variables. Indeed, if $f : [-1,1] \times \Re \longrightarrow [-1,1]$ is Lipiński's function from Example 1.7 then $f|[-1,1] \times [k, k+1]$ is almost continuous for any integer k (see Theorem 4.6 below) but f is not almost continuous.

Theorem 2.2 *If $f : I \longrightarrow \Re^k$ is almost continuous and A is a subset of I then $f|A$ is almost continuous.*

P r o o f . By Lemma 2.2 it is sufficient to prove that $f|cl_A(J)$ is almost continuous for any component J of $int(A)$. If $cl_A(J)$ is compact then, by Theorem 2.1, $f|cl_A(J)$ is almost continuous. Otherwise, $cl_A(J)$ can be represented as a union of countably many of compact intervals satisfying the assumptions of Lemma 2.3. Thus, almost continuity of $f|cl_A(J)$ follows from that lemma.

Q.E.D.

On the other hand it is easy to find a set $A \subset I$ and a continuous function $f : A \longrightarrow \Re$ which cannot be extended to an almost continuous real function defined on the entire interval I.

Theorem 2.3 *For any non-void subset A of I and positive integer k the following conditions are equivalent:*

(*i*) *each almost continuous function $f : A \longrightarrow \Re^k$ can be extended to an almost continuous function $f^* : I \longrightarrow \Re^k$,*

(*ii*) *each continuous function $f : A \longrightarrow \Re^k$ can be extended to an almost continuous function $f^* : I \longrightarrow \Re^k$,*

(*iii*) *the set $I \setminus A$ is bilaterally c-dense in itself,*

(*iv*) *there exists a function $g : I \setminus A \longrightarrow \Re^k$ such that $f \cup g$ is almost continuous for each almost continuous function $f : A \longrightarrow \Re^k$.*

P r o o f . Obviously only two implications need to be proved.

$(ii) \Longrightarrow (iii)$. Assume that x_0 is a point of $I \setminus A$ and $card((x_0, x_0 + \varepsilon) \setminus A) < 2^\omega$ for some positive ε. We define a function $f : A \longrightarrow \Re^k$ by $f(x) = (0, 0, \ldots, 0)$ for $x < x_0$ and $f(x) = (1/(x - x_0), 0, \ldots, 0)$ for $x > x_0$. Then f is continuous and it has no Darboux extension on whole interval I.

$(iii) \Longrightarrow (iv)$. Let $(J_n)_n$ be a sequence of all components of $int(A)$. Note that $\overline{J_n} \subset A$ for each $n \in N$. Let $(F_\alpha)_{\alpha<2^\omega}$ be a sequence of all minimal blocking sets $F \subset I \times \Re^k$ such that $dom(F_\alpha) \subset \overline{J_n}$ for no $n \in N$. Then $card(dom(F_\alpha) \setminus A) = 2^\omega$ for every $\alpha < 2^\omega$. We choose $(x_\alpha, y_\alpha) \in F_\alpha$ such that $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Put $g(x) = y_\alpha$ if $x = x_\alpha$, $\alpha < 2^\omega$ and $g(x) = 0$ for other $x \in I \setminus A$.

Let $f : A \longrightarrow \Re^k$ be an arbitrary almost continuous function. Then $f \cup g$ is almost continuous, too. Indeed, let F be a minimal blocking set in $I \times \Re^k$. Then either $dom(F_\alpha) \subset \overline{J_n}$ for some $n \in N$ or $F = F_\alpha$ for some $\alpha < 2^\omega$. In the first case F is blocking in $\overline{J_n} \times \Re^k$ and therefore $f \cap F \neq \emptyset$. Otherwise $(x_\alpha, y_\alpha) \in F \cap g$. Thus $F \cap (f \cup g) \neq \emptyset$ and consequently $f \cup g$ is almost continuous.

Q.E.D.

The following simple but useful fact is proved in [30] (for $k = 1$).

Theorem 2.4 *Assume that $h : (a, b) \longrightarrow \Re^k$ is almost continuous and $y, z \in \Re^k$, $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$ and $h_3 = h_1 \cup h_2$. Then h_1, h_2, h_3 are almost continuous iff $y \in C^+(h, a)$, $z \in C^-(h, b)$ and $y \in C^+(h, a)$, $z \in C^-(h, b)$ respectively.*

Theorem 2.5 *For any non-empty subset A of I and positive integer k the following conditions are equivalent:*

- (i) *each bounded almost continuous function $f : A \longrightarrow \Re^k$ can be extended to an almost continuous function $f^* : I \longrightarrow \Re^k$,*
- (ii) *any bounded continuous function $f : A \longrightarrow \Re^k$ can be extended to an almost continuous function $f^* : I \longrightarrow \Re^k$,*
- (iii) *the set $I \setminus A$ is c-dense in itself.*

P r o o f . $(ii) \Longrightarrow (iii)$. Assume that $x_0 \in I \setminus A$ and $card((x_0 - \varepsilon, x_0 + \varepsilon) \setminus A) < 2^\omega$ for some positive ε. Then the function $g : A \longrightarrow \Re^k$ given by

$g(x) = (0, \ldots, 0)$ for $x < x_0$ and $g(x) = (1, 0, \ldots, 0)$ for $x > x_0$ is continuous and it has no Darboux extension on whole I.

$(iii) \Longrightarrow (i)$. Let $f : A \longrightarrow \Re^k$ be a bounded almost continuous function and J be a k-dimensional closed cube containing $f(A) \cup \{0\}$. Fix $t \in J$. Let $(J_n)_n$ be a sequence of all components of $int(A)$. It follows from Theorem 2.4 that for each $n \in N$ the function $f|J_n$ can be extended to an almost continuous function $f_n^* : \overline{J_n} \longrightarrow J$. Let $(F_\alpha)_{\alpha<2^\omega}$ be a sequence of all minimal blocking sets in $I \times J$. As in the proof of Theorem 2.5 we choose a sequence of points $(x_\alpha, y_\alpha)_{\alpha<2^\omega}$ such that $(x_\alpha, y_\alpha) \in F_\alpha$ for each α and $\{(x_\alpha, y_\alpha) : \alpha < 2^\omega\}$ is a function which agrees with f on the set A. Put $f^*(x) = f_n^*(x)$ for $x \in \overline{J_n}$ and $n \in N$, $f^*(x_\alpha) = y_\alpha$ for $x = x_\alpha$, $\alpha < 2^\omega$ and $f^*(x) = 0$ otherwise. Then $f^*|A = f$ and $f^* \in \mathcal{A}(I, J)$. From Theorem 1.9,(1) we obtain that $f^* \in \mathcal{A}(I, \Re^k)$.

The implication $(i) \Longrightarrow (ii)$ is obvious.

Q.E.D.

3 Compositions.

Obviously the class $\mathcal{D}(\Re, \Re)$ of all functions having the Darboux property is closed under compositions. Thus the following fact follows from Theorem 1.10,(1).

Theorem 3.1 *The composition $g \circ f$ of almost continuous functions $f, g : \Re \longrightarrow \Re$ has Darboux property.*

On the other hand, there exists a function $f \in \mathcal{A}(I, I)$ such that $f \circ f$ has no fixed point and consequently is not almost continuous [39] (see also [33], where for arbitrary positive integers n, m almost continuous functions $f : I^n \longrightarrow I^m$, $g : I^m \longrightarrow I^n$ are constructed such that the composition $g \circ f$ has no fixed point). The foregoing suggests also the following question.

Problem 3.1 *Is any Darboux function from $\Re$ into $\Re$ a composition of (two) almost continuous functions ? [39], [49]*

Theorem 3.2 *Assume $A(c)$. Then any function from the class $\mathcal{D}^*(\Re, \Re)$ is the composition of two almost continuous functions [49].*

Now we shall prove a similar result concerning (K, G) pairs of topological spaces.

Proposition 3.1 *Suppose that X is a T_1 space, (X, Y) and (Y, Z) are (K, G) pairs with blocking families $\mathcal{F}$ and $\mathcal{K}$, respectively. If $card(\mathcal{F}) = card(\mathcal{K}) = card(X) = card(Y) = \kappa$ and any $F \in \mathcal{F}$ satisfies the following condition:*

(1) *the set $dom(F)$ cannot be decomposed into less than κ subsets which are nowhere dense in $dom(F)$,*

then every function $f : X \longrightarrow Z$ such that

(2) *$card(G \cap f^{-1}(z)) = \kappa$ for any $F \in \mathcal{F}$, $z \in Z$ and any non-empty set G open in $dom(F)$,*

can be expressed as a composition of two almost continuous functions $f_1 \in \mathcal{A}(X, Y)$ and $f_2 \in \mathcal{A}(Y, Z)$.

P r o o f . Let $(x_\alpha)_{\alpha<\kappa}$, $(F_\alpha)_{\alpha<\kappa}$ and $(K_\alpha)_{\alpha<\kappa}$ be sequences of all points of X, and all sets from $\mathcal{F}$ and $\mathcal{K}$, respectively. We choose for each $\alpha < \kappa$ points $(a_\alpha, a'_\alpha) \in F_\alpha$, $(b_\alpha, b'_\alpha) \in K_\alpha$ and $c_\alpha \in Y$ such that

(i) $a_\alpha \neq a_\beta$, $b_\alpha \neq b_\beta$ and $c_\alpha \neq c_\beta$ for $\alpha \neq \beta$,

(ii) if $a'_\alpha = a'_\beta$ for $\alpha, \beta < \kappa$, then $f(a_\alpha) = f(a_\beta)$,

(iii) if $a'_\alpha = b_\beta$ for $\alpha, \beta < \kappa$, then $f(a_\alpha) = b'_\beta$,

(iv) if $a'_\alpha = c_\beta$ for $\alpha, \beta < \kappa$, then $f(a_\alpha) = f(x_\beta)$,

(v) $b_\alpha \neq c_\beta$ for $\alpha, \beta < \kappa$.

We shall verify that it is possible to choose such points. Assume that $\alpha < \kappa$ and (a_β, a'_β), (b_β, b'_β), c_β are chosen for $\beta < \alpha$. Fix for each $x \in dom(F_\alpha)$ a point $y(x)$ such that $(x, y(x)) \in F_\alpha$. Put $A_\beta = \{x \in dom(F_\alpha) : y(x) = a'_\beta\}$, $B_\beta = \{x \in dom(F_\alpha) : y(x) = b_\beta\}$, $C_\beta = \{x \in dom(F_\alpha) : y(x) = c_\beta\}$ for $\beta < \alpha$. Now, if $D = dom(F_\alpha) \setminus \bigcup_{\beta<\alpha}(A_\beta \cup B_\beta \cup C_\beta \cup \{a_\beta\})$ has cardinality κ, we choose any $a_\alpha \in D$ and put $a'_\alpha = y(a_\alpha)$. Otherwise, $int_{dom(F_\alpha)}(cl_{dom(F_\alpha)}(A_\beta \cup B_\beta \cup C_\beta))$ is non-void for some $\beta < \alpha$. Let, e.g., $G = int_{dom(F_\alpha)}(cl_{dom(F_\alpha)} A_\beta) \neq \emptyset$. Then $G \times \{a'_\beta\} \subset F_\alpha$, so $G \subset A_\beta$. Choose $a_\alpha \in G \cap f^{-1}(f(a_\beta)) \setminus \{a_\gamma : \gamma <$

$\alpha\}$ and put $a'_\alpha = a'_\beta$. Next we choose $b_\alpha \in dom(K_\alpha)\setminus(\{b_\beta, a'_\beta : \beta < \alpha\}\cup\{a'_\alpha\})$ and $c_\alpha \in Y \setminus (\{b_\beta, a'_\beta, c_\beta : \beta < \alpha\} \cup \{a'_\alpha, b_\alpha\})$. Let f_1, f_2 be defined by

$$f_1(x) = \begin{cases} a'_\alpha & \text{for } x = a_\alpha,\ \alpha < \kappa, \\ c_\alpha & \text{for } x = x_\alpha \text{ and } x \notin \{a_\beta : \beta < \kappa\} \end{cases}$$

$$f_2(y) = \begin{cases} f(a_\alpha) & \text{for } y = a'_\alpha,\ \alpha < \kappa \\ b'_\alpha & \text{for } y = b_\alpha,\ \alpha < \kappa \\ f(x_\beta) & \text{for } y = c_\beta,\ \beta < \kappa \\ f(x_0) & \text{otherwise.} \end{cases}$$

Then $f_1 \in \mathcal{A}(X,Y)$, $f_2 \in \mathcal{A}(Y,Z)$ and $f = f_2 \circ f_1$.

Q.E.D.

Now we shall consider under which conditions for f_1 and f_2 the composed map $f_2 \circ f_1$ is almost continuous.

Theorem 3.3 *For each $f \in \mathcal{A}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$ the composed map $g \circ f$ is almost continuous [60].*

Theorem 3.4 *If $h : X \longrightarrow Y$ is a homeomorphism and $f : Y \longrightarrow Z$ is almost continuous then the composition $f \circ h$ is almost continuous [27].*

Proof. Let $G \subset X \times Z$ be an open neighbourhood of $f \circ h$. Then $G_0 = \{(h(x), z) : (x, z) \in G\}$ is an open neighbourhood of the function f in $Y \times Z$. Let $g : Y \longrightarrow Z$ be a continuous function contained in G_0. Then $g \circ h : X \longrightarrow Z$ is a continuous function contained in G.

Q.E.D.

Corollary 3.1 *Suppose that h is a homeomorphic injection from X into Y such that $rng(h)$ is closed in Y. Then $f \circ h \in \mathcal{A}(X,Z)$ for any $f \in \mathcal{A}(Y,Z)$.*

Proof. It follows from Theorem 2.1 that $f|rng(h) \in \mathcal{A}(rng(h), Z)$. Since h is a homeomorphism between X and $rng(h)$, $f \circ h = (f|rng(h)) \circ h \in \mathcal{A}(X,Z)$.

Q.E.D.

Theorem 3.5 *If a space X is compact, Y is a Hausdorff space, $g \in \mathcal{C}(X,Y)$ and $f \in \mathcal{A}(Y,Z)$, then $f \circ g \in \mathcal{A}(X,Z)$.*

P r o o f . This theorem is proved in [60]. We give here only the sketch of the proof which is based on the notion of blocking sets. Suppose that $f \circ g$ is not almost continuous. Let K be a blocking set for $f \circ g$ in $X \times Z$. Then $\{(g(x), z) : (x, z) \in K\}$ is a blocking set for $f|rng(g)$, which contradicts Theorem 2.1.

Q.E.D.

Note that the assumption about X is important. Indeed, let $f : [-1,1] \times \Re \longrightarrow [-1,1]$ be Lipiński's function from Example 1.7 and let $g : [-1,1] \times \Re \longrightarrow [-1,1] \times \{0\}$ be a continuous function given by $g(x,y) = (x,0)$. Then f is not almost continuous, $f|([-1,1] \times \{0\})$ is almost continuous (by Corollary 3.1) and $f = (f|[-1,1] \times \{0\}) \circ g$.

Theorem 3.6 *If A is a subspace of $\Re$, $f \in \mathcal{C}(A, \Re)$ and $g \in \mathcal{A}(\Re, \Re^k)$ then $g \circ f \in \mathcal{A}(A, \Re^k)$.*

P r o o f . This is a consequence of Theorem 3.5 if A is a compact interval, of Lemma 2.3 if A is an interval and, finally, of Lemma 2.2 for arbitrary subset A of $\Re$.

Q.E.D.

Lemma 3.1 *Suppose that C is a closed, dense in itself and nowhere dense subset of I and $f : I \longrightarrow \Re$ satisfies the following conditions:*

(1) $rng(f)$ is an interval,

(2) $f|J$ is almost continuous for any component J of the complement of C,

(3) both unilateral cluster sets of the function f at the end-points of components of the set $I \setminus C$ equal $rng(f)$.

Then, f is almost continuous.

P r o o f . Suppose that f is not almost continuous. Let K be a minimal blocking set for f in $I \times rng(f)$. Conditions (2) and (3) and Theorem 2.4 imply that $f|\overline{J}$ is almost continuous, for arbitrary component J of $I \setminus C$. Therefore, $dom(K)$ is contained in the closure of no component of $I \setminus C$ and consequently there exists a component J contained in $dom(K)$. Suppose $J = (s,t)$. Since $(K|[0,s])$ and $(K|[t,1])$ are not blocking in $I \times rng(f)$, there are continuous functions $g, h : I \longrightarrow rng(f)$ such that $(g|[0,s]) \cap$

$K = \emptyset = (h|[t,1]) \cap K$. Finally it is easy to observe that the function $k = g|[0,s] \cup f|(s,t) \cup h|[t,1]$ is almost continuous and disjoint from K, a contradiction.

Q.E.D.

Theorem 3.7 *Let $f_1 \in \mathcal{A}(I,\Re)$, $f_2 \in \mathcal{A}(\Re,\Re)$, the set D of all points at which f_1 is not continuous is nowhere dense and adequate unilateral cluster sets of the function f_1 at the end-points of components of the set $I \setminus \overline{D}$ coincide with $rng(f_1)$. Then $f_2 \circ f_1$ is almost continuous.*

Proof. By Theorem 3.6 we obtain that $(f_2 \circ f_1)|J \in \mathcal{A}(J,\Re)$ for any component J of $I \setminus \overline{D}$. Since $f_2 \circ f_1$ has the Darboux property, $rng(f_2 \circ f_1)$ is an interval. Note that unilateral cluster sets of the function $f_2 \circ f_1$ at the end-points of components of the set $I \setminus \overline{D}$ equal $rng(f_2 \circ f_1)$. Almost continuity of the composition $f_2 \circ f_1$ now follows from Lemmas 3.1 and 2.3.

Q.E.D.

Lemma 3.2 *Let $\mathcal{F}_0, \mathcal{K}_0$ be families of subsets of X and Y respectively such that $max(card(\mathcal{F}_0), card(\mathcal{K}_0)) \leq \kappa$ and $card(M) \geq \kappa \geq \omega$ for all $M \in \mathcal{F}_0 \cup \mathcal{K}_0$. Then for every injection $f : X \longrightarrow Y$ there exist sets $A, C \subset X$ and $D \subset Y$ such that:*

(1) A, C and $f^{-1}(D)$ are pairwise disjoint,,

(2) $card(A \cap F) = \kappa$ for each $F \in \mathcal{F}_0$ and $card(K \setminus (f(A) \cup f(C) \cup D)) \geq \kappa$ for each $K \in \mathcal{K}_0$,

(3) $card(C) = \kappa$ and $card(D) = \kappa$.

Proof. Let $(F_\alpha)_{\alpha<\kappa}$, $(K_\alpha)_{\alpha<\kappa}$ be sequences of sets from classes $\mathcal{F}_0$ and $\mathcal{K}_0$ respectively, such that $card(\{\alpha : F_\alpha = F\}) = \kappa$ for each $F \in \mathcal{F}_0$ and $card(\{\alpha : K_\alpha = K\}) = \kappa$ for each $K \in \mathcal{K}_0$. Choose sequences $(a_\alpha)_{\alpha<\kappa}$, $(b_\alpha)_{\alpha<\kappa}$, $(c_\alpha)_{\alpha<\kappa}$ and $(d_\alpha)_{\alpha<\kappa}$ of points such that the following conditions hold for each $\alpha < \kappa$:

(*i*) $a_\alpha, c_\alpha \in F_\alpha \setminus (\{a_\beta, c_\beta\} \cup f^{-1}(\{b_\beta, d_\beta : \beta < \alpha\}))$ and $a_\alpha \neq c_\alpha$,

(*ii*) $b_\alpha, d_\alpha \in K_\alpha \setminus (\{b_\beta, d_\beta : \beta < \alpha\} \cup \{f(a_\beta), f(c_\beta) : \beta \leq \alpha\})$ and $b_\alpha \neq d_\alpha$.

Put $A = \{a_\alpha : \alpha < \kappa\}$, $B = \{b_\alpha : \alpha < \kappa\}$, $C = \{c_\alpha : \alpha < \kappa\}$ and $D = \{d_\alpha : \alpha < \kappa\}$. Then the conditions (1) and (3) are obvious. Since $\{a_\alpha : F_\alpha = F\} \subset A \cap F$, $card(A \cap F) = \kappa$ for all $F \in \mathcal{F}_0$. Similarly, for each $K \in \mathcal{K}_0$ we have $\{b_\alpha : K_\alpha = K\} \subset K \setminus (f(A) \cup f(C) \cup D)$ and therefore $card(K \setminus (f(A) \cup f(C) \cup D)) \geq \kappa$.

Q.E.D.

Proposition 3.2 *Suppose that (X, Y) and (Y, Z) are (K, G) pairs with blocking families $\mathcal{F}$ and $\mathcal{K}$, respectively, $card(Y) = card(\mathcal{F}) = card(\mathcal{K}) = \kappa \geq \omega$ and $card(Z) \leq \kappa$. If a function $f : X \longrightarrow Y$ satisfies the following condition: $card(f(dom(F))) = \kappa$ for each $F \in \mathcal{F}$, then for every surjection $g : Y \longrightarrow Z$ there exist almost continuous surjections $h_1 : Y \longrightarrow Z$ and $h_2 : X \longrightarrow Y$ such that $h_1 \circ f = g \circ h_2$.*

P r o o f . Let $(F_\alpha)_{\alpha<\kappa}$ and $(K_\alpha)_{\alpha<\kappa}$ be sequences of all sets from the classes $\mathcal{F}$ and $\mathcal{K}$, respectively. Let $(y_\alpha)_{\alpha<\kappa}$ and $(z_\alpha)_{\alpha<\kappa}$ be sequences of all points of Y and Z, respectively (the sequence $(z_\alpha)_\alpha$ may not be one-to-one). Let $\sim$ be the equivalence relation in X induced by f, i.e. $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. The equivalence class of x with respect to relation $\sim$ is denoted by $[x]$. For $A \subset X$ let $A^\sim = \{[x] : x \in A\}$ and let $f^\sim : X^\sim \longrightarrow Y$ be defined by $f^\sim([x]) = f(x)$. Let $\mathcal{F}_0 = \{(dom(F))^\sim : F \in \mathcal{F}\}$ and $\mathcal{K}_0 = \{dom(K) : K \in \mathcal{K}\}$. Note that all assumptions of Lemma 3.2 are satisfied for $f^\sim$, $\mathcal{F}_0$ and $\mathcal{K}_0$. Let $A, C \subset X^\sim$ and D be as in that lemma. Moreover, let $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ be partitions of the sets A and $B = Y \setminus (f^\sim(A \cup C) \cup D)$ into subsets which intersect each set from $\mathcal{F}_0$ and $\mathcal{K}_0$, respectively and let $h_C : C \longrightarrow Y$ and $h_D : D \longrightarrow Z$ be arbitrary surjections. Now we define surjections $h_1 : Y \longrightarrow Z$ and $h_2 : X^\sim \longrightarrow Y$ such that $h_1 \circ f^\sim = g \circ h_2^\sim$.

(*a*) $h_2^\sim | C = h_C$ and $h_1 | D = h_D$.

(*b*) Let $[x] \in A$. Then $[x] \in A_\alpha$ for some $\alpha < \kappa$. If $[x] \in A_\alpha \cap (dom(F_\alpha))^\sim$, then we choose $y \in Y$ such that $(s, y) \in F_\alpha$ for some $s \in [x]$ and define $h_2^\sim([x]) = y$. If $[x] \in A_\alpha \setminus (dom(F_\alpha))^\sim$, we put $h_2^\sim([x]) = y_\alpha$.

(*c*) Let $y \in B$. Then $y \in B_\alpha$ for some $\alpha < \kappa$. If $y \in B_\alpha \cap dom(K_\alpha)$, we choose $z \in Z$ such that $(y, z) \in K_\alpha$ and define $h_1(y) = z$. If $y \in B_\alpha \setminus dom(K_\alpha)$, put $h_1(y) = z_\alpha$.

(*d*) If $[x] \notin A \cup C$ then $f^\sim([x]) \in (B \cup D)$. Since g is a surjection, there exists $y \in Y$ such that $g(y) = h_1(f^\sim([x]))$ and we define $h_2^\sim([x]) = y$.

(e) If $y \in f^{\sim}(A \cup C)$ then $y = f^{\sim}([x])$ for exactly one $x \in A \cup C$ and we put $h_1(y) = g(h_2^{\sim}([x]))$.

One can verify that the definition of and $h_2^{\sim}$ is correct and $h_1 \circ f^{\sim} = g \circ h_2^{\sim}$. Now define $h_2 : X \longrightarrow Y$ by $h_2(x) = h_2^{\sim}([x])$. Then h_2 is a surjection and $h_1 \circ f = g \circ h_2$. Moreover, h_2 and h_1 intersect all blocking sets from $\mathcal{F}$ and $\mathcal{K}$, respectively, so they are almost continuous.

Q.E.D.

Corollary 3.2 *For any bijection $b : I^n \longrightarrow I^m$ there exist almost continuous surjections $h : I^m \longrightarrow I^m$ and $k : I^n \longrightarrow I^n$ for which the compositions $b \circ k$ and $h \circ b$ are almost continuous.*

P r o o f . Using Proposition 3.2 for $f = b$ and $g = id_{I^m}$ we obtain almost continuous surjections $h : I^m \longrightarrow I^m$ and $h_2 : I^n \longrightarrow I^m$ such that $h_2 = h \circ b$. Similarly, for $f = id_{I^n}$ and $g = b$ there exist almost continuous surjections $h_1 : I^n \longrightarrow I^m$ and $k : I^n \longrightarrow I^n$ such that $h_1 = b \circ k$. The functions h and k have the required properties.

Q.E.D.

For a given family $\mathcal{F}$ of functions from X into X we define two classes:

$\mathcal{M}_i(\mathcal{F})$ — the class of all function $f : X \longrightarrow X$ such that $g \circ f \in \mathcal{F}$ for any g from $\mathcal{F}$,

$\mathcal{M}_o(\mathcal{F})$ — the class of all function $f : X \longrightarrow X$ such that $f \circ g \in \mathcal{F}$ for any g from $\mathcal{F}$.

Problem 3.2 *Characterize the classes $\mathcal{M}_o(\mathcal{A}(I, I))$ and $\mathcal{M}_i(\mathcal{A}(I, I))$.*

Finally remark that there exist a continuous surjection f from I onto I and $g \notin \mathcal{A}(I, I)$ such that $g \circ f \in \mathcal{A}(I, I)$ [41].

4 Cartesian products and diagonals.

Theorem 4.1 *Assume that X_2 is a compact space, $f_1 \in \mathcal{A}(X_1, Y_1)$ and $f_2 \in \mathcal{C}(X_2, Y_2)$. Then the cartesian product $h = (f_1, f_2) : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ of f_1 and f_2 (given by $h(x_1, x_2) = (f_1(x_1), f_2(x_2))$) is almost continuous (cf. [1] if all X_1, X_2, Y_1, Y_2 are compact).*

P r o o f . Suppose that $K \subset X_1 \times X_2 \times Y_1 \times Y_2$ is a blocking set for h. We shall verify that $F = \{(x_1, y_1) \in X_1 \times Y_1 : (x_1, x_2, y_1, f_2(x_2)) \in K$ for some $x_2 \in X_2\}$ is blocking for f_1 in $X_1 \times Y_1$.

(1) F is closed. Indeed, fix $(x_1, y_1) \in X_1 \times Y_1 \setminus F$. Then for each $x_2 \in X_2$, $(x_1, x_2, y_1, f_2(x_2)) \notin K$. For every $x_2 \in X_2$ choose open neighbourhoods $U_1(x_2)$ of x_1, $U_2(x_2)$ of x_2, $V_1(x_2)$ of y_1 and $V_2(x_2)$ of $f(x_2)$ such that $U_1(x_2) \times U_2(x_2) \times V_1(x_2) \times V_2(x_2)$ is disjoint with K. Let $W(x_2) = U_2(x_2) \cap f_2^{-1}(V_2(x_2))$. Then $U_1(x_2) \times W(x_2) \times V_1(x_2) \times V_2(x_2) \subset X_1 \times X_2 \times Y_1 \times Y_2 \setminus K$ is an open neighbourhood of the point $(x_1, x_2, y_1, f(x_2))$. Let $W(t_1), \ldots, W(t_n)$ be a finite subcovering of X_2 chosen from the covering $\{W(x_2) : x_2 \in X_2\}$. Denote $U = \bigcap_{i=1}^n U_1(t_i)$ and $V = \bigcap_{i=1}^n V_1(t_i)$. Then $U \times V$ is an open neighbourhood of (x_1, y_1) disjoint with F.

(2) Since K and h are disjoint, F is disjoint with f_1.

(3). If $g : X_1 \longrightarrow Y_1$ is continuous then $(g, f_2) : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ is continuous, too. Since K is blocking, $(x_1, x_2, g(x_1), f_2(x_2)) \in K$ for some $x_1 \in X_1$, $x_2 \in X_2$, and therefore $(x_1, g(x_1)) \in F$.

Q.E.D.

Note that the assumption about X_2 is important. Indeed, let $X_1 = Y_1 = Y_2 = [-1, 1]$, $X_2 = \Re$, $f_0 : [-1, 1] \longrightarrow [-1, 1]$ be the function from Example 1.1, f be Lipiński's function from Example 1.7 and $f_1 \equiv 0$. Suppose that $h = (f_0, f_1)$ is almost continuous. Since f is a composition of h and the projection π_1 from $[-1, 1] \times [-1, 1]$ into $[-1, 1]$, Theorem 3.1 implies almost continuity of f, a contradiction.

Theorem 4.2 *Let $\mathcal{M}_p(\mathcal{A}(I, I))$ be the class of all functions f from I into $\Re$ such that $(f, g) \in \mathcal{A}(I \times I, \Re \times \Re)$ when $g \in \mathcal{A}(I, \Re)$. Then $\mathcal{M}_p(\mathcal{A}(I, I)) = \mathcal{C}(I, \Re)$.*

P r o o f . The inclusion " $\supset$" follows from Theorem 4.1. Now assume that $f : I \longrightarrow \Re$ is not almost continuous. It will be proved in Theorem 6.2 that there exists $g \in \mathcal{A}(I, \Re)$ such that $f + g \notin \mathcal{A}(I, \Re)$. Suppose that $(f, g) \in \mathcal{A}(I \times I, \Re^2)$. Then $f + g$, as the composition (f, g) with the "addition" map is almost continuous, a contradiction.

Q.E.D.

Now we shall consider functions f, g defined on the same space. Assume that $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$. The map $f \Delta g : X \longrightarrow Y \times Z$ defined by $f \Delta g(x) = (f(x), g(x))$ for any $x \in X$ is called a *diagonal* of f and g. It is obvious that $f\Delta g = (f,g) \circ d$, where $d : X \longrightarrow \{(x,x) : x \in X\}$ is given by $d(x) = (x,x)$. The following fact follows from Corollary 3.1.

Theorem 4.3 *If X is a Hausdorff space and $(f,g) \in \mathcal{A}(X \times X, Y \times Z)$ then $f\Delta g \in \mathcal{A}(X, Y \times Z)$.*

Theorem 4.4 *If $f \in \mathcal{A}(X,Y)$ and $g \in \mathcal{C}(X,Z)$ then $f\Delta g \in \mathcal{A}(X, Y \times Z)$ [30].*

P r o o f . If X is compact, this theorem follows from Theorems 4.1 and 4.3. In the case of metric spaces X, Y and Z it is proved in [48] (see also [1] for X, Y, Z metric and compact).

In the general case assume that $f\Delta g$ is not almost continuous. Let K be a blocking set for $f\Delta g$ in $X \times (Y \times Z)$. It is easy to verify that $F = \{(x,y) : (x,y,g(x)) \in K\}$ is blocking for f in $X \times Y$.

Q.E.D.

For arbitrary topological spaces X, Y, Z let $\mathcal{M}_d(\mathcal{A}(X, Y \times Z)$ be the family of all functions from X into Y such that $f\Delta g$ is almost continuous provided $g : X \longrightarrow Z$ is almost continuous. As in Theorem 4.2 one can prove the following equality.

Corollary 4.1 $\mathcal{M}_d(\mathcal{A}(\Re, \Re \times \Re)) = \mathcal{C}(\Re, \Re)$

Lemma 4.1 *Suppose that D is a closed and nowhere dense subset of I, $(I_n)_n$ is a sequence of all components of the complement of D and $f : I \longrightarrow \Re^k$ satisfies the following conditions:*

(1) $f|\overline{I_n}$ is almost continuous for every $n \in N$,

(2) $f|D$ is continuous.

Then f is almost continuous.

P r o o f . We can assume that $0, 1 \in D$. Let G be an open neighbourhood of f in $I \times \Re^k$. For each $x \in D$ we choose open intervals U_x, V_x such that:

(a) $(x, f(x)) \in U_x \times V_x \subset \overline{U_x} \times \overline{V_x} \subset G$,

(b) $f|(D \cap \overline{U_x}) \subset \overline{U_x} \times V_x$,

(c) $inf(U_x) < inf(D \cap \overline{U_x}) \leq sup(D \cap \overline{U_x}) < sup(U_x)$ (this condition must be interpreted unilaterally at the points 0 and 1).

Since $f|D$ is compact, there are points $x_1, \ldots, x_n \in D$ such that $f|D \subset \bigcup_{i=1}^{n}(U_{x_i} \times V_{x_i})$. We can assume that $0 \in U_{x_1}$, $1 \in U_{x_n}$ and $inf(U_{x_i}) < inf(U_{x_j})$ for $i < j$. If $U_{x_i} \cap U_{x_{i+1}} \neq \emptyset$ then there exists a continuous function g defined on $W = U_{x_i} \cup U_{x_{i+1}}$ such that $g \subset G$ and $g(x) = f(x)$ for $x \in \{inf(D \cap W), sup(D \cap W)\}$. Let $W_1, \ldots, W_m$ be components of the union $\bigcup_{i=1}^{n} U_{x_i}$. For every $i = 1, \ldots, m$ there exists a continuous function g_{2i-1} defined on W_i such that $g_{2i-1} \subset G$ and $g(x) = f(x)$ for $x \in \{inf(D \cap W_i), sup(D \cap W_i)\}$. Additionally, for $i < m$ there exists n_i such that $I_{n_i} = (sup(D \cap W_i), inf(D \cap W_{i+1}))$. Since $f|\overline{I_{n_i}}$ is almost continuous, there exists a continuous function $g_{2i} : \overline{I_{n_i}} \longrightarrow \Re^k$ such that $g_{2i} \subset G$, and $g_{2i}(x) = f(x)$ for $x \in \{inf(I_{n_i}), sup(I_{n_i})\}$. Then $\bigcup_{i=1}^{2m-1} g_i$ is a continuous function defined on all of I and contained in G.

Q.E.D.

Theorem 4.5 *Suppose that f_1, f_2 are almost continuous real functions defined on I and D is the set of points at which f_1 is discontinuous. If $f_1|\overline{D}$ is continuous and $\overline{D} \subset C(f_2)$, then $f_1 \Delta f_2$ is almost continuous.*

P r o o f . This is a consequence of Lemma 4.1 and Theorem 4.4.

Note that the assumption "$\overline{D} \subset C(f_2)$" is important. Indeed, let $f_1, f_2 : [-1,1] \longrightarrow [-1,1]$ be defined by $f_i(x) = (-1)^i sin(1/x)$ for $x \neq 0$, $i = 1,2$ and $f_1(x) = f_2(x) = 1$. Suppose that $f_1 \Delta f_2 \in \mathcal{A}([-1,1],[-1,1]^2)$. Then, as in Theorem 4.2, $f_1 + f_2 \in \mathcal{A}([-1,1],[-1,1])$, but this is impossible because $f_1 + f_2$ does not have the Darboux property.

Theorem 4.6 *Suppose that X_2 is compact, $f_1 \in \mathcal{A}(X_1, Y)$, $f_2 \in \mathcal{C}(X_2, Y)$, and $F \in \mathcal{C}(Y \times Y, Y)$. Then the function $F(f_1, f_2) : X_1 \times X_2 \longrightarrow Y$ defined by $F(f_1, f_2)(x_1, x_2) = F(f_1(x_1), f_2(x_2))$ for $(x_1, x_2) \in X_1 \times X_2$ is almost continuous.*

P r o o f . The function $(f_1, f_2) : X_1 \times X_2 \longrightarrow Y \times Y$ is almost continuous by Theorem 4.1. Hence $F(f_1, f_2)$ is almost continuous by Theorem 3.1.

Q.E.D.

Corollary 4.2 *If X_2 is compact, $f_1 \in \mathcal{A}(X_1, Y)$ and $f_2 \in \mathcal{C}(X_2, Y)$, then*

(1) $F : X_1 \times X_2 \longrightarrow Y$ given by $F(x_1, x_2) = f_1(x_1)$ is almost continuous,

(2) if $Y = \Re$ then $F_1(x_1, x_2) = f_1(x_1) + f_2(x_2)$, $F_2(x_1, x_2) = f_1(x_1)f_2(x_2)$, $F_3(x_1, x_2) = max(f_1(x_1), f_2(x_2))$ and $F_4(x_1, x_2) = min(f_1(x_1), f_2(x_2))$ are almost continuous.

Note that the assumption about X_2 in the last results is important (see e.g. Lipiński's function from Example 1.7). As it was remarked by Grande [26], continuity of all sections of $f : I \times I \longrightarrow I$ does not imply almost continuity of f.

Example 4.1 *There exists a function $f : I \times I \longrightarrow I$ such that f_x, f^y are continuous for each $x, y \in I$ but f is not almost continuous .*

Indeed, let $f : I \times I \longrightarrow I$ be defined by $f(x,y) = 2xy/(x^2 + y^2)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then all sections of f are continuous but for a connected set $D = \{(x,x) : x \in I\}$ we have $f(D) = \{0,1\}$. Thus the function f_0 from I into I given by $f_0(x) = f(x,x)$ does not have Darboux property. Suppose that f is almost continuous. Then $f|D$ is almost continuous, in contradiction with Corollary 3.1.

Lemma 4.2 *Assume that $m \in N$, $F \in \mathcal{C}(\Re^2, \Re)$, $f \in \mathcal{A}(\Re^m, \Re)$, $g \in \mathcal{C}(\Re, \Re)$ and $h : \Re^{m+1} \longrightarrow \Re$ is defined by*

$$h(x_1, \ldots, x_m, x_{m+1}) = F(f(x_1, \ldots x_m), g(x_{m+1})).$$

If there exists a compact subset K of $\Re$ such that $[h \neq 0] \subset \Re^m \times K$ then h is almost continuous.

P r o o f . Fix reals a, b such that $K \subset (a,b)$ and an open neighbourhood $G \subset \Re^{m+2}$ of h. Let $(S_k)_k$ be a sequence of all m-dimensional cubes of the form $\prod_{i=1}^{m}[k_i, k_i + 1]$, where $k_1, \ldots, k_m$ are integers. For each $k \in N$ choose positive reals r_k, q_k such that $S_k \times [a - r_k, a + r_k] \times [-r_k, r_k] \subset G$ and $S_k \times [b - q_k, b + q_k] \times [-q_k, q_k] \subset G$. By Theorem 4.6, $h|\Re^m \times [a,b]$ is almost continuous and therefore there exists a continuous function $t : \Re^m \times [a,b] \longrightarrow \Re$ contained in $G \setminus \bigcup_{k=1}^{\infty}(S_k \times \{a\} \times ((-\infty, -r_k] \cup [r_k, \infty)) \cup S_k \times \{b\} \times ((-\infty, -q_k] \cup [q_k, \infty)))$. Let t_a be a surface consisting of all closed segments in $\Re^{m+2}$ with end-points $(x, a, t(x,a))$ and $(x, a - |t(x,a)|, 0)$ for all $x \in \Re^m$. Analogously, let t_b be a surface consisting of all closed segments in $\Re^{m+2}$ with end-points $(x, b, t(x,b))$ and $(x, b + |t(x,b)|, 0)$ for all $x \in \Re^m$. Then one can easily see that $t \cup t_a \cup t_b \cup (\Re^{m+1} \setminus dom(t \cup t_a \cup t_b)) \times \{0\}$ is a continuous function contained in G. Q.E.D.

Corollary 4.3 *If $f \in \mathcal{A}(\Re^m, \Re)$, $g \in \mathcal{C}(\Re, \Re)$ and the support of g is bounded then the function $h : \Re^{m+1} \longrightarrow \Re$, defined by*

$$h(x_1, \ldots x_{m+1}) = f(x_1, \ldots x_m) \cdot g(x_{m+1}),$$

is almost continuous.

Theorem 4.7 *Each almost continuous function $f : \Re^k \longrightarrow \Re$ can be extended to almost continuous function $f^* : \Re^{k+1} \longrightarrow \Re$ such that $f^*(x, 0) = f(x)$ for all $x \in \Re^k$ (cf. [37], Theorem 5.6.).*

P r o o f . Put $g(x) = max(1 - |x|, 0)$ for $x \in \Re$ and $f^*(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_k) \cdot g(x_{k+1})$. The almost continuity of f^* follows from Corollary 4.3. Moreover, $f^*(x, 0) = f(x)$ for all $x \in \Re^k$.

Q.E.D.

Corollary 4.4 *Assume that k, m are positive integers and $k < m$. Then each almost continuous function $f : \Re^k \longrightarrow \Re$ can be extended to an almost continuous function $f^* : \Re^m \longrightarrow \Re$ such that $f^*(x_1, \ldots, x_k, 0, \ldots, 0) = f(x_1, \ldots x_k)$ for $(x_1, \ldots, x_k) \in \Re^k$.*

5 Limits of sequences.

Lemma 5.1 *Suppose that (X, Y) is a (K, G) pair, $\mathcal{F}$ is a blocking family for (X, Y) and $max(\omega, \kappa) \leq \lambda = card(\mathcal{F})$. Then there exists a partition of X into κ many sets X_α $(\alpha < \kappa)$, such that $card(dom(F) \cap X_\alpha) \geq \lambda$ for each $\alpha < \kappa$ and $F \in \mathcal{F}$.*

P r o o f . Let $(F_\alpha)_{\alpha<\lambda}$ be a sequence of all sets from the family $\mathcal{F}$, let $\varphi : \lambda \longrightarrow \kappa \times \lambda \times \lambda$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be an arbitrary bijection. For each $\alpha < \lambda$ we choose $x_\alpha \in dom(F_{\varphi_3(\alpha)}) \setminus \{x_\beta : \beta < \alpha\}$. Then the sets $X_\alpha = \{x_\beta : \beta < \lambda, \varphi_1(\beta) = \alpha\}$ for $0 < \alpha < \kappa$ and $X_0 = X \setminus \bigcup_{0<\alpha<\kappa} X_\alpha$ form the required partition.

Q.E.D.

Recall that a function $f : X \longrightarrow Y$ is a discrete limit of a net $(f_\sigma)_{\sigma\in\Sigma}$, where $(\Sigma, \leq)$ is a directed set, iff for each $x \in X$ there exists $\sigma_0 \in \Sigma$ such that $f_\sigma(x) = f(x)$ whenever $\sigma_0 \leq \sigma$.

Proposition 5.1 *Suppose that (X, Y) is (K, G) pair, $\mathcal{F}$ is a blocking family for (X, Y) and $(\Sigma, \leq)$ is a directed set such that $card(\mathcal{F}) \geq card(\Sigma) \geq \omega$. Then each function $f : X \longrightarrow Y$ is a discrete limit of a net of almost continuous functions from X into Y.*

P r o o f . Let $card(\mathcal{F}) = \lambda$ and $\mathcal{F} = \{F_\alpha : \alpha < \lambda\}$. By Lemma 5.1 there is a partition $\{X_\sigma : \sigma \in \Sigma\}$ of X such that $card(dom(F) \cap X_\sigma) \geq \lambda$ for every $\sigma \in \Sigma$ and $F \in \mathcal{F}$. For each $\sigma \in \Sigma$ and $\alpha < \lambda$ choose $(x_{\sigma,\alpha}, y_{\sigma,\alpha}) \in F_\alpha$ such that $x_{\sigma,\alpha} \in X_\sigma \setminus \{x_{\sigma,\beta} : \beta < \alpha\}$. Let f_σ be defined by $f_\sigma(x_{\sigma,\alpha}) = y_{\sigma,\alpha}$ for $\alpha < \lambda$ and $f_\sigma(x) = f(x)$ for $X \setminus \{x_{\sigma,\alpha} : \alpha < \lambda\}$. Then any f_σ is almost continuous and for every $x \in X$ there exists $\sigma_0 \in \Sigma$ such that $f_\sigma(x) = f(x)$ for all $\sigma \geq \sigma_0$.

Q.E.D.

Corollary 5.1 *Suppose that (X, Y) is (K, G) pair with an infinite blocking family $\mathcal{F}$. Then each function $f : X \longrightarrow Y$ is a discrete limit of a sequence of almost continuous functions in $X \times Y$.*

In particular each function $f : \Re \longrightarrow \Re$ is a discrete limit of a sequence of almost continuous functions $(f_n)_n$ [34].

Remark 5.1 *If $f : \Re \longrightarrow \Re$ is Lebesgue measurable (has the Baire property), then f is a discrete limit of a sequence of measurable functions (with the Baire property) from the class $\mathcal{A}(\Re, \Re)$ [26].*

Recall the following notion. A sequence $(f_\alpha)_{\alpha<\omega_1}$ of functions from X into Y converges to a function $f : X \longrightarrow Y$ if for each $x \in X$ and each neighbourhood U of $f(x)$ there exists $\alpha < \omega_1$ such that $f_\beta(x) \in U$ for all $\alpha < \beta < \omega_1$ [57].

Corollary 5.2 *Suppose that (X, Y) is (K, G) pair and $\mathcal{F}$ is an uncountable blocking family for (X, Y). Then each function $f : X \longrightarrow Y$ is a limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of almost continuous functions in $X \times Y$.*

In particular every function $f : \Re \longrightarrow \Re$ is a limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of almost continuous functions.

Remark 5.2 *Suppose $A(c)$ $(A(m))$. If $f : \Re \longrightarrow \Re$ is measurable (has the Baire property) then it is a transfinite limit of a sequence of measurable functions (with the Baire property) from the class $\mathcal{A}(\Re, \Re)$ (see [26]).*

Suppose that Y is a metric space and $\mathcal{F}$ is an arbitrary family of functions from X into Y. The class of all limits of uniformly convergent sequences of functions from $\mathcal{F}$ will be denoted by $\overline{\mathcal{F}}$. Note that:

(1) The class $\mathcal{A}(\Re,\Re)$ is not closed with respect to uniform limits [34], [38].

(2) $\overline{\mathcal{A}(\Re,\Re)} \subset \overline{\mathcal{D}(\Re,\Re)} = \mathcal{U}$, where the class $\mathcal{U}$ is defined in [13].

(3) There exists a connectivity function f from I into I which is not a limit of uniformly convergent sequence of almost continuous functions [29]. Thus $\mathcal{U} \setminus \overline{\mathcal{A}(\Re,\Re)} \neq \emptyset$.

Suppose that (X,Y) is a (K,G) pair with a blocking family $\mathcal{F}$, (Y,ρ_Y) is a metric space and κ_Y is the least cardinal for which there exists a family of κ_Y many sets of the first category in Y which union is of the second category (or $\kappa_Y = 0$ if Y is of the first category on itself). For arbitrary $f : X \longrightarrow Y$ and positive ε we define an ε-hull $S(f,\varepsilon)$ of f in $X \times Y$ as $S(f,\varepsilon) = \{(x,y) \in X \times Y : \ \rho_Y(f(x),y) < \varepsilon\}$. We define two conditions for f:

(α) for sufficiently small $\varepsilon > 0$ and for every blocking set $K \in \mathcal{F}$ either $card(dom(K \cap S(f,\varepsilon))) \geq card(\mathcal{F})$ or $B_Y(f(x),\varepsilon) \subset K_x$ for some $x \in X$,

(β) for each $\varepsilon > 0$ and for every blocking set $K \in \mathcal{F}$ either $card(dom(K \cap S(f,\varepsilon))) \geq \kappa_Y$ or $int_Y(K \cap S(f,\varepsilon))_x \neq \emptyset$ for some $x \in X$.

Under the assumptions and denotations above the following implications hold.

Proposition 5.2

(1) For every function f from X into Y we have:

$$(\alpha) \Longrightarrow f \in \overline{\mathcal{A}(X,Y)}$$

(2) Moreover, if $(Y,+)$ is a topological group and it is a Baire space then

$$f \in \overline{\mathcal{A}(X,Y)} \Longrightarrow (\beta)$$

P r o o f . (1) For sufficiently small positive ε we shall find an almost continuous function g from X into Y contained in $S(f,\varepsilon)$. Let $card(\mathcal{F}) = \lambda$ and let $(K_\alpha)_{\alpha<\lambda}$ be a sequence of all blocking sets from $\mathcal{F}$. For each $\alpha < \lambda$ we can choose a point $(x_\alpha, y_\alpha) \in K_\alpha \cap S(f,\varepsilon)$ such that for $\alpha, \beta < \lambda$ the condition $x_\alpha = x_\beta$ implies $y_\alpha = y_\beta$. Indeed, assume that (x_β, y_β) are chosen for $\beta < \alpha$. There are two possible cases. If $card(dom(K_\alpha \cap S(f,\varepsilon))) \geq \lambda$ then we choose $(x_\alpha, y_\alpha) \in K_\alpha \cap S(f,\varepsilon)$ such that $x_\alpha \neq x_\beta$ for all $\beta < \alpha$. In the other case, $B_Y(f(x),\varepsilon) \subset (K_\alpha)_x$ for some $x \in X$ and we put $x_\alpha = x$ and $y_\alpha = y_\beta$ whenever $x = x_\beta$ for some $\beta < \alpha$ or $y_\alpha = f(x)$ otherwise. It is easy to verify that the function $g : X \longrightarrow Y$ defined by $g(x_\alpha) = y_\alpha$ for $\alpha < \lambda$ and $g(x) = f(x)$ for other x is almost continuous and $g \subset S(f,\varepsilon)$.

(2) Suppose that $(f_n)_n$ is a uniformly convergent sequence of almost continuous functions and f is the limit of $(f_n)_n$. Fix $K \in \mathcal{F}$, a positive ε and suppose that $card(dom(K \cap S(f,\varepsilon))) < \kappa_Y$. Then $f_n \subset S(f,\varepsilon/2)$ for some positive integer n. Additionally there exists a positive δ such that $f_n + y \subset S(f,\varepsilon)$ whenever $y \in B_Y(0,\delta)$. By Theorems 4.4 and 3.3, $f_n + y \in \mathcal{A}(X,Y)$ for any $y \in B_Y(0,\delta)$. Thus $f_n + y$ intersects K, i.e.

$$\forall y \in B_Y(0,\delta) \quad \exists (x_y, t_y) \in K \cap (f_n + y) \subset S(f,\varepsilon).$$

Since $card(dom(K \cap S(f,\varepsilon))) < \kappa_Y$, the set $A = \{y \in B_Y(0,\delta) : x_y = x\}$ is of the second category in $B_Y(0,\delta)$ for some $x \in X$. Then $(x, t_y) \in f_n + y$ for $y \in A$ and therefore, $t_y = f_n(x) + y$. Thus the set $\{t_y : y \in A\}$ is of the second category in $f_n(x) + B_Y(0,\delta)$ and consequently there exists a non-empty open set $U \subset B_Y(0,\delta)$ such that $f_n(x) + U \subset cl(\{t_y : y \in A\})$. Since K is closed, $f_n(x) + U \subset K_x$ and we obtain (β).

Q.E.D.

Corollary 5.3

(1) If for sufficiently small positive ε and for every blocking set K in $\Re^2$ either $card(dom(K \cap S(f,\varepsilon))) = 2^\omega$ or $(f(x) - \varepsilon, f(x) + \varepsilon) \subset K_x$ for some $x \in \Re$ then $f \in \overline{\mathcal{A}(\Re,\Re)}$.

(2) Assume $A(c)$. If $f \in \overline{\mathcal{A}(\Re,\Re)}$ then for each positive ε and blocking set K in $\Re^2$ either $card(dom(K \cap S(f,\varepsilon))) = 2^\omega$ or $int((K \cap S(f,\varepsilon))_x) \neq \emptyset$ for some $x \in X$.

Corollary 5.4 *Every function $f : I \longrightarrow \Re$ which satisfies the condition:*

(*) *card*$(\{x \in J : \ |f(x) - q| \leq \varepsilon\}) = 2^\omega$ *for each subinterval* $J \subset I$*, rational* q *and positive* ε *,*

is a limit of uniformly convergent sequence of almost continuous functions. In particular, $\mathcal{D}^* \subset \overline{\mathcal{A}(I, \Re)}$.

P r o o f . By Proposition 5.2 it is sufficient to verify that f satisfies condition (α). We shall prove that $card(dom(S(f, \varepsilon))) = 2^\omega$ for every blocking set $F \subset I \times \Re$, positive ε and f satisfying the condition (*). Indeed, fix $n \in N$ such that $2/n < \varepsilon$. For every integer k define $F_k = \{x \in I : \ \exists y \in \Re \ (x, y) \in F$ and $|y - (2k-1)/n| \leq 1/n\} = dom(F \cap (I \times [(2k-2)/n, 2k/n]))$. Note that each F_k is closed and the interior of the set $\bigcup_{k \in Z} F_k = dom(F)$ is non-empty (see Theorem 1.2 (3)). Hence there exists a non-degenerate interval J which is contained in F_{k_0} for some integer k_0. Put $m = 2k_0 - 1$ and $A = \{x \in J : |f(x) - m/n| \leq 1/n\}$. By (*), $card(A) = 2^\omega$. Moreover, for each $x \in A$ there exists y_x such that $(x, y_x) \in F$ and $|y_x - m/n| \leq 1/n$. Hence $|f(x) - y_x| \leq 2/n < \varepsilon$ for $x \in A$ and the condition (α) holds.

Q.E.D.

Problem 5.1 *Characterize the class of all uniform limits of almost continuous functions from* I^k *into* I *[34].*

Note that the analogous problem is open for the class $Conn(I, I)$ [11]. For $k > 1$ the class $Conn(I^k, I)$ is closed under this operation [25]. This is not true for the class $\mathcal{A}(I^k, I)$.

Example 5.1 *For any* k *there exists a uniformly convergent sequence of almost continuous functions from* I^k *into* I *which limit is not almost continuous.*

Indeed, let $(f_n)_n$ be a uniformly convergent sequence of almost continuous functions from I into I which limit f is not almost continuous. Let g_n, g be functions from I^k into I defined by $g_n(x_1, \ldots, x_k) = f_n(x_1)$ and $g(x_1, \ldots, x_k) = f(x_1)$. Then g is a uniform limit of g_n, by Corollary 4.2 all g_n are almost continuous and, by Theorem 2.1, g is not almost continuous.

Now we shall consider the notion of almost continuous approximation which was introduced in [1]. A sequence $(f_n)_n$ of functions from X into Y almost continuously approximates a function $f : X \longrightarrow Y$ if for every sequence $(x_n)_n$ of points from X, either there exists n such that $f_n(x_n) =$

$f(x_n)$ or there exists a subsequence (x_{n_i}) of (x_n) and $x \in X$ such that $x_{n_i} \longrightarrow x$ and $f_{n_i}(x_{n_i}) \longrightarrow f(x)$ (here X and Y are metric) [1].

Theorem 5.1 *The sequence $(f_n)_n$ almost continuously approximates f iff for each open neighbourhood U of f there exists $n \in N$ such that $f_n \in U$ [1].*

Corollary 5.5 *If $(f_n)_n$ is a sequence of functions from the class $\mathcal{A}(X,Y)$ and $(f_n)_n$ almost continuously approximates f, then $f \in \mathcal{A}(X,Y)$ [1].*

Theorem 5.2 *Assume that X and Y are compact metric spaces. Then $f \in \mathcal{A}(X,Y)$ iff there exists a sequence $(f_n)_n$ of continuous functions which approximates almost continuously f [1].*

6 Operations.

6.1 Sums.

Proposition 6.1 *Suppose that $(Y,+)$ is a topological group, (X,Y) is a (K,G) pair, $\mathcal{K}$ is a blocking family for (X,Y) and κ is a cardinal such that $max(\omega,\kappa) \leq \lambda = card(\mathcal{K})$. Then for any family $\mathcal{F}$ of functions from X into Y with $card(\mathcal{F}) = \kappa$ the following condition holds:*

$U_a(\mathcal{F})$: there exists $g : X \longrightarrow Y$ such that $g + f \in \mathcal{A}(X,Y)$ for all $f \in \mathcal{F}$.

In particular, each function f from X into Y can be expressed as a sum of two almost continuous functions in $X \times Y$.

P r o o f . Let $\{X_\alpha : \alpha < \kappa\}$ be a partition of the space X such that $card(dom(K) \cap X_\alpha) \geq \lambda$ for each $\alpha < \kappa$ and $K \in \mathcal{K}$ (such partition exists by Lemma 5.1). Let $(K_\beta)_{\beta<\lambda}$ be a sequence of all blocking sets from the family $\mathcal{K}$. For every $\alpha < \kappa$ and $\beta < \lambda$ choose $(x_{\alpha,\beta}, y_{\alpha,\beta}) \in K_\beta$ such that $x_{\alpha,\beta} \in X_\alpha \setminus \{x_{\alpha,\gamma} : \gamma < \beta\}$. Let $g : X \longrightarrow Y$ be defined by $g(x_{\alpha,\beta}) = y_{\alpha,\beta} - f_\alpha(x_{\alpha,\beta})$ for $\alpha < \kappa$ and $\beta < \lambda$ and $g(x) = 0$ otherwise (0 denotes the neutral element of the group $(Y,+)$). Since $(x_{\alpha,\beta}, y_{\alpha,\beta}) \in (g + f_\alpha) \cap K_\beta$ for $\beta < \lambda$, $g + f_\alpha \in \mathcal{A}(X,Y)$.

Now assume that $f_0 \equiv 0$. For an arbitrary function $f : X \longrightarrow Y$ and the family $\mathcal{F} = \{f, f_0\}$ let g be a function such that $h = g + f \in \mathcal{A}(X,Y)$ and $g + f_0 \in \mathcal{A}(X,Y)$. Then $f = (-g) + h$, $g \in \mathcal{A}(X,Y)$ and by Theorem 3.3, $-g \in \mathcal{A}(X,Y)$.

Q.E.D.

Corollary 6.1 *If $\mathcal{F}$ is a family of functions from $\Re$ into $\Re$ and $card(\mathcal{F}) \le 2^{\omega}$ then $U_a(\mathcal{F})$ holds. In particular, any function f from $\Re$ into $\Re$can be written a sum of two almost continuous functions f_1, f_2 [34].*

Remark 6.1 *If a function $f : \Re \longrightarrow \Re$ is Lebesgue measurable (has the Baire property) then it can be represented as a sum of two almost continuous functions which are measurable (have the Baire property) [26].*

The foregoing results suggest the question of how "big" can be families $\mathcal{F}$ for which the condition $U_a(\mathcal{F})$ holds. For arbitrary topological space X and topological group $(Y,+)$ let $a(X,Y)$ denote the least cardinal κ for which there exists a family $\mathcal{F}$ of functions from X into Y such that $card(\mathcal{F}) = \kappa$ and $U_a(\mathcal{F})$ is false (or $a(X,Y) = 0$ if the condition $U_a(Y^X)$ holds). Note that Proposition 6.1 implies the inequality $a(X,Y) > card(\mathcal{K})$ for any (K,G) pair (X,Y) with blocking family $\mathcal{K}$. In particular, $a(\Re,\Re) > 2^{\omega}$. Additionally, it is easy to see that the condition $U_a(\Re^{\Re})$ is false. Indeed, for every function $g : \Re \longrightarrow \Re$ there exists a function f such that $f + g$ does not have the Darboux property. Therefore $a(\Re,\Re) \ne 0$. Hence the assumption $(2^{\omega})^+ = 2^{2^{\omega}}$ (which is a consequence of the Generalized Continuum Hypothesis for example) implies the equality $a(\Re,\Re) = 2^{2^{\omega}}$.

Problem 6.1 *Can the equality $a(\Re,\Re) = 2^{2^{\omega}}$ be proved in ZFC?*

Now we shall prove the condition $U_a(\mathcal{F})$ for some families of real functions of the power $2^{2^{\omega}}$. Suppose that κ is a cardinal, $\mathcal{I}$ is a fixed family of subsets of I and $\mathcal{F}$ is a fixed family of real functions defined on I. We shall say that $\mathcal{F}$ is $(\mathcal{I},\kappa)$ regular if there exists a subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $card(\mathcal{F}_0) = \kappa$ and for each $f \in \mathcal{F}$ there exists $f_0 \in \mathcal{F}_0$ with $[f \ne f_0] \in \mathcal{I}$. A family $\mathcal{I}$ of subsets of I has the property (B) if:

(1) if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$,

(2) if $A \in \mathcal{I}$ then $J \setminus A$ includes a non-empty perfect set for every subinterval J of I.

Lemma 6.1 *Assume that $\mathcal{F}$ is a family of real functions defined on I and $card(\mathcal{F}) = 2^{\omega}$. Then there exists a function g such that for each $f \in \mathcal{F}$ and minimal blocking set K, $dom(K \cap (f+g))$ intersects every non-empty perfect set contained in $dom(K)$.*

P r o o f . Let $\mathcal{F} = \{f_\alpha : \alpha < 2^\omega\}$, let $\{K_\beta : \beta < 2^\omega\}$ be the family of all minimal blocking sets in $I \times \Re$ and let $\varphi : 2^\omega \longrightarrow 2^\omega \times 2^\omega \times 2^\omega$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ be an arbitrary bijection. For $\beta < 2^\omega$ arrange all nonempty perfect subsets of $dom(K_\beta)$ in a sequence $(F_{\beta,\gamma})_{\gamma<2^\omega}$. For each $\alpha < 2^\omega$ choose $(x_\alpha, y_\alpha) \in K_{\varphi_2(\alpha)}$ such that $x_\alpha \in F_{(\varphi_2(\alpha),\varphi_3(\alpha))} \setminus \{x_\gamma : \gamma < \alpha\}$. Then the function g defined by $g(x_\alpha) = y_\alpha - f_{\varphi_1(\alpha)}(x_\alpha)$ for $\alpha < 2^\omega$ and $g(x) = 0$ for other x satisfies the conditions of the lemma.

Q.E.D.

Theorem 6.1 *Assume that $\mathcal{I}$ is a family of subsets of I with the property (B) and $\mathcal{F}$ is an $(\mathcal{I}, 2^\omega)$ regular family of real functions defined on I. Then the condition $U_a(\mathcal{F})$ holds.*

P r o o f . Let $\mathcal{F}_0$ be a subfamily of $\mathcal{F}$ such that $card(\mathcal{F}_0) = 2^\omega$ and for each $f \in \mathcal{F}$ there exists $f_0 \in \mathcal{F}_0$ such that $[f \neq f_0] \in \mathcal{I}$. Fix $f \in \mathcal{F}$ and $f_0 \in \mathcal{F}_0$ such that $[f \neq f_0] \in \mathcal{I}$. Let g be the function defined in Lemma 6.1 for the family $\mathcal{F}_0$. Then $(g + f) \cap K \neq \emptyset$ for any blocking K. Indeed, suppose that $(g + f) \cap K = \emptyset$. Then $C = dom((g + f_0) \cap K) \subset [f \neq f_0]$ and therefore $C \in \mathcal{I}$. Thus $dom(K) \setminus C$ includes a non-empty perfect set, in contradiction with the choice of g.

Q.E.D.

Corollary 6.2 *Let $\mathcal{F}$ be the family of all Lebesgue measurable functions (all functions with the Baire property) from $\Re$ into $\Re$, $\mathcal{F}_0$ be the family of Borel measurable functions and $\mathcal{I}$ be the ideal of measure zero (of the first category) subsets of $\Re$. Then there exists a function g from $\Re$ into $\Re$ such that $f + g \in \mathcal{A}(\Re, \Re)$ for each $f \in \mathcal{F}$.*

For arbitrary families $\mathcal{X}, \mathcal{Y}$ of real functions defined on a topological space X let $\mathcal{M}_a(\mathcal{X}, \mathcal{Y})$ denote the maximal additive class of $\mathcal{X}$ with respect to $\mathcal{Y}$, i.e.

$$\mathcal{M}_a(\mathcal{X}, \mathcal{Y}) = \{f \in \mathcal{X} : f + g \in \mathcal{Y} \text{ for each } g \in \mathcal{X}\}.$$

We shall write $\mathcal{M}_a(\mathcal{X})$ instead of $\mathcal{M}_a(\mathcal{X}, \mathcal{X})$ and call this family the *maximal additive class* of $\mathcal{X}$.

Theorem 6.2

$$\mathcal{M}_a(\mathcal{A}(\Re, \Re), \mathcal{Y}) = \mathcal{C}(\Re, \Re)$$

whenever $\mathcal{Y} \in \{\mathcal{A}(\Re, \Re), \mathcal{C}onn(\Re, \Re), \mathcal{D}(\Re, \Re)\}$.

P r o o f . For $\mathcal{Y} = \mathcal{A}(\Re, \Re)$ see [30]. The same arguments work for other $\mathcal{Y}$.
Q.E.D.

Theorem 6.3 *For any positive integer k we have*

$$\mathcal{M}_a(\mathcal{A}(\Re^k, \Re)) = \mathcal{C}(\Re^k, \Re).$$

P r o o f . This equality follows for $k = 1$ from Theorem 6.2. For any k the inclusion $\mathcal{C}(\Re^k, \Re) \subset \mathcal{M}_a(\mathcal{A}(\Re^k, \Re))$ follows from Theorems 4.4 and 3.3. Now assume that a function $g : \Re^k \longrightarrow \Re$ is discontinuous at a point $x_0 \in \Re^k$. Let h be a homeomorphic injection of $\Re$ into $\Re^k$ such that $rng(h)$ is closed in $\Re^k$, $h(0) = x_0$, $g \circ h$ is discontinuous at 0 and there exists a homeomorphism $h_1 : \Re^k \longrightarrow \Re^k$ such that $h_1(x, 0, \ldots, 0) = h(x)$ for $x \in \Re$. Let $f_0 : \Re \longrightarrow \Re$ be an almost continuous function such that $f_0 + g \circ h \notin \mathcal{A}(\Re, \Re)$. By Theorem 4.7, there exists an almost continuous extension $f_1 : \Re^k \longrightarrow \Re$ of f_0 such that $f_1(x, 0, \ldots, 0) = f_0(x)$ for any $x \in \Re$. By Theorem 3.4, $f = f_1 \circ h_1^{-1}$ is almost continuous. Suppose that $f + g$ is almost continuous. Then $(f + g)|h(\Re)$ is almost continuous (by Theorem 2.1), and therefore, $(f+g) \circ h \in \mathcal{A}(\Re, \Re)$. But $(f+g) \circ h = f \circ h + g \circ h = f_0 + g \circ h$, a contradiction. Thus $g \notin \mathcal{M}_a(\mathcal{A}(\Re^k, \Re))$.
Q.E.D.

Corollary 6.3 *For any positive integers k and m,*

$$\mathcal{M}_a(\mathcal{A}(\Re^k, \Re^m)) = \mathcal{C}(\Re^k, \Re^m).$$

P r o o f . The inclusion $\mathcal{C}(\Re^k, \Re^m) \subset \mathcal{M}_a(\mathcal{A}(\Re^k, \Re^m))$ follows from Theorems 4.4 and 3.3. Assume that a function $g : \Re^k \longrightarrow \Re^m$, $g = (g_1, \ldots, g_m)$, is discontinuous at a point $x_0 \in \Re^k$. Then g_i is discontinuous at x_0 for some $i \leq m$. By Theorem 6.3, $f + g_i$ is not almost continuous for some almost continuous function f from $\Re^k$ into $\Re$. By Theorem 4.4, the function $h = (h_1, \ldots, h_m) : \Re^k \longrightarrow \Re^m$, where $h_i = f$ and $h_j \equiv 0$ for $j \neq i$, is almost continuous. Observe that $\pi_i \circ (h + g) = f + g_i$ (where π_i denotes the projection onto i^{th} axis) is not almost continuous and, by Theorem 3.3, $h + g$ is not almost continuous.

Q.E.D.

6.2 Products.

Proposition 6.2 *Suppose that F is a topological field, (X,F) is a (K,G) pair with an infinite blocking family $\mathcal{K}$ and $k>1$. Then each function $f: X \longrightarrow F$ can be expressed as a scalar product of two almost continuous functions $f_1, f_2 : X \longrightarrow F^k$ (i.e. $f=\sum_{i=1}^{k} f_{1,i}\cdot f_{2,i}$, where $f_1=(f_{1,1},\ldots,f_{1,k})$ and $f_2=(f_{2,1},\ldots,f_{2,k})$).*

Proof. By Proposition 6.1 $f: X \longrightarrow F$ can be expressed as a sum of almost continuous functions $g_1, g_2 : X \longrightarrow F$. Now define $f_1, f_2 : X \longrightarrow F^k$ in the following way: $f_1(x)=(g_1(x),1,0,\ldots,0)$ and $f_2(x)=(1,g_2(x),0,\ldots,0)$ for $x \in X$. By Theorem 4.4 f_1 and f_2 are almost continuous and, clearly, $f=f_1\cdot f_2$.

Q.E.D.

Corollary 6.4

(1) for each $m \in N$, $k>1$ and $f: I^m \longrightarrow \Re$ there exist $f_1, f_2 \in \mathcal{A}(I^m, \Re^k)$ such that $f=f_1\cdot f_2$.

(2) for each $k>1$ and $f:\Re \longrightarrow \Re$ there exist $f_1, f_2 \in \mathcal{A}(\Re,\Re^k)$ such that $f=f_1\cdot f_2$.

Note that the condition above is false for $k=1$. Indeed, it is well-known that a function $f:\Re \longrightarrow \Re$ may not be a product of Darboux functions [45] and therefore, of almost continuous functions. J. Ceder proved in [16] that a function $f:\Re \longrightarrow \Re$ is a product of two Darboux functions iff it possesses the following property:

(JC) : f has a zero in each subinterval in which it changes sign.

In particular, if $rng(f) \subset (0,\infty)$ or $rng(f) \subset (-\infty,0)$ then f is a product of two Darboux functions.

Theorem 6.4 *Suppose $A(c)$. A real function f defined on $\Re$ is a product of two almost continuous functions iff it has the property (JC) [48].*

Proposition 6.3 *Suppose that $(X,\Re)$ is a (K,G) pair, $\mathcal{K}$ is a blocking family for $(X,\Re)$ and κ is a cardinal such that $max(\omega,\kappa) \le \lambda = card(\mathcal{K})$. If $\mathcal{F}$ is a family of real functions defined on X, $card(\mathcal{F})=\kappa$ and $rng(f) \subset (-\infty,0)$ or $rng(f) \subset (0,\infty)$ for all $f \in \mathcal{F}$, then there exists a function $g: X \longrightarrow (0,\infty)$ such that $g\cdot f$ is almost continuous for each $f \in \mathcal{F}$.*

P r o o f . By Proposition 6.1 there exists a function $g_0 : X \longrightarrow \Re$ such that $g_0 + h \in \mathcal{A}(X, \Re)$ for any $h \in \{ln \circ |f| : f \in \mathcal{F}\}$. Put $g = exp(g_0)$. Then $g(x) > 0$ for each $x \in X$ and for every $f \in \mathcal{F}$ we have:

$$g \cdot f = sgn(f) \cdot exp(g_0) \cdot exp \circ ln \circ |f| = sgn(f) \cdot exp \circ (g_0 + ln \circ |f|) \in \mathcal{A}(X, \Re).$$

Q.E.D.

Corollary 6.5 *If $(X, \Re)$ is a (K, G) pair with an infinite blocking family and f is an arbitrary function from X into $(0, \infty)$ then there exist almost continuous functions $f_1, f_2 : X \longrightarrow (0, \infty)$ such that $f = f_1 \cdot f_2$. In particular, every function $f : \Re \longrightarrow (0, \infty)$ can be expressed as a product of two almost continuous functions [26].*

For an arbitrary family $\mathcal{F}$ of real functions defined on a topological space X let us define the following condition:

$U_m(\mathcal{F})$: there exists a non-zero function $g : X \longrightarrow \Re$ such that $f \cdot g \in \mathcal{A}(X, \Re)$ whenever $f \in \mathcal{F}$.

Theorem 6.5 *Suppose $A(c)$. Then every family $\mathcal{F}$ of real functions defined on $\Re$ with $card(\mathcal{F}) < 2^\omega$ satisfies the condition $U_m(\mathcal{F})$ [50].*

Example 6.1 *Let $\mathcal{F}$ be the family of all characteristic functions of singletons and $g : \Re \longrightarrow \Re$ be a function such that $f \cdot g \in \mathcal{A}(\Re, \Re)$ for all $f \in \mathcal{F}$. Then $g \equiv 0$ [50].*

For an arbitrary topological space X let $m(X, \Re)$ denote the least cardinal κ for which there exists a family $\mathcal{F}$ of real functions from X such that $card(\mathcal{F}) = \kappa$ and $U_m(\mathcal{F})$ is false (or $m(X, \Re) = 0$ if $U_m(\Re^X)$ holds).

Corollary 6.6 *$A(c)$ implies the equality $m(\Re, \Re) = 2^\omega$.*

Problem 6.2 *Can the equality $m(\Re, \Re) = 2^\omega$ be proved in ZFC ?*

For arbitrary families $\mathcal{X}, \mathcal{Y}$ of real functions defined on a topological space X let $\mathcal{M}_m(\mathcal{X}, \mathcal{Y})$ denote the maximal multiplicative class of $\mathcal{X}$ with respect to $\mathcal{Y}$, i.e.

$$\mathcal{M}_m(\mathcal{X}, \mathcal{Y}) = \{f \in \mathcal{X} : f \cdot g \in \mathcal{Y} \text{ for all } g \in \mathcal{X}\}.$$

We shall write $\mathcal{M}_m(\mathcal{X})$ instead of $\mathcal{M}_m(\mathcal{X}, \mathcal{X})$ and call this family the *maximal multiplicative class* of $\mathcal{X}$.

For arbitrary interval Y of $\Re^m$ let us define the family $\mathcal{M}(I, Y)$ of all functions $f : I \longrightarrow Y$ having the following property: if x_0 is a right-hand (left-hand) side point of discontinuity of f, then $f(x_0) = 0$ and there is a sequence $(x_n)_n$ converging to x_0 such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$. If X is any space then $\mathcal{M}(X, Y)$ denotes the class of all functions $f : X \longrightarrow Y$ such that $f \circ h \in \mathcal{M}(I, Y)$ for any homeomorphic injection $h : I \longrightarrow X$. This class was introduced by Fleissner [21] (for $X = Y = \Re$).

Theorem 6.6

$$\mathcal{M}_m(\mathcal{A}(\Re, Z), \mathcal{Y}) = \mathcal{M}(\Re, Z)$$

whenever $\mathcal{Y} \in \{\mathcal{A}(\Re, Z), Conn(\Re, Z), \mathcal{D}(\Re, Z)\}$ *and* $Z \in \{\Re, [0, \infty)\}$.

P r o o f . For $Z = \Re$ and $\mathcal{Y} = \mathcal{A}(\Re, \Re)$ see [30]. The proof is analogous for other Z and $\mathcal{Y}$.

Q.E.D.

The similar theorem can be considered for scalar products of functions with values in $\Re^k$.

Theorem 6.7

(1) Suppose that $Z \in \{\Re, [0, \infty)\}$, $g \in \mathcal{M}_m(\mathcal{A}(\Re^k, Z^n))$ *and* $g = (g_1, \ldots, g_n)$. *Then:*

(1.1) $g_i \in \mathcal{M}_m(\mathcal{A}(\Re^k, Z))$ *for every* $i = 1, \ldots, n$,

(1.2) $C(g) \subset [g = 0]$,

(1.3) *if* $n = 1$ *then* $g \in \mathcal{M}(\Re^k, \Re)$.

(2) Moreover, if $Z = [0, \infty)$, *then:*

(2.1) $\mathcal{M}_m(\mathcal{A}(\Re^k, Z^n)) \subset \mathcal{M}(\Re^k, Z^n)$,

(2.2) $\mathcal{M}_m(\mathcal{A}(\Re, Z^n)) = \mathcal{M}(\Re, Z^n)$.

(3) If $Z = (0, \infty)$ *then* $\mathcal{M}_m(\mathcal{A}(\Re^k, Z^n)) = \mathcal{C}(\Re^k, Z^n)$.

P r o o f . (1.1) Suppose that $g_i \notin \mathcal{M}_m(\mathcal{A}(\Re^k, Z))$ for some $i \leq m$. Then there exists $h \in \mathcal{A}(\Re^k, Z)$ such that $g_i \cdot h \notin \mathcal{A}(\Re^k, Z)$. By Theorem 4.4 the function $f : \Re^k \longrightarrow \Re^n$, $f = (f_1, \ldots, f_n)$, where $f_i = h$ and $f_j \equiv 0$ for $j \neq i$, is almost continuous and $f \cdot g = h \cdot g_i$ is not almost continuous, contrary to $g \in \mathcal{M}_m(\mathcal{A}(\Re^k, \Re^n))$.

(1.2) Suppose that g is discontinuous at x_0. Then g_t is discontinuous at x_0 for some $t \leq n$. By (1.1), $g_i(x_0) = 0$ if g_i is discontinuous at x_0. Assume that $g_i(x_0) \neq 0$ for some $i \leq n$. Then g_i is continuous at x_0. Consequently $g_t + g_i$ is discontinuous at x_0 and $(g_t + g_i)(x_0) \neq 0$. Therefore $g_t + g_i \notin \mathcal{M}_m(\mathcal{A}(\Re^k, \Re))$ and $(g_t + g_i) \cdot h \notin \mathcal{A}(\Re^k, \Re)$ for some $h \in \mathcal{A}(\Re^k, \Re)$. By Theorems 4.4 and 3.3 the function $f : \Re^k \longrightarrow \Re^n$, $f = (f_1, \ldots, f_n)$ defined by $f_t = f_i = h$ and $f_j \equiv 0$ for $j \notin \{t, i\}$, is almost continuous and $g \cdot f = (g_t + g_i) \cdot h \notin \mathcal{A}(\Re^k, \Re)$, a contradiction.

(1.3) Assume that $n = 1$ and $g \in \mathcal{M}_m(\mathcal{A}(\Re^k, Z)) \setminus \mathcal{M}(\Re^k, Z)$. Let $h : \Re \longrightarrow \Re^k$ be a homeomorphic injection such that $g \circ h \notin \mathcal{M}(\Re, Z)$, $rng(h)$ is closed in $\Re^k$ and there exists a homeomorphism $h_1 : \Re^k \longrightarrow \Re^k$ such that $h_1(x, 0, \ldots, 0) = h(x)$ for $x \in \Re$. Then $f_0 \cdot (g \circ h) \notin \mathcal{A}(\Re, \Re)$ for some $f_0 \in \mathcal{A}(\Re, \Re)$. By Theorem 4.7, there exists an almost continuous extension $f_1 : \Re^k \longrightarrow \Re$ of f_0 such that $f_1(x, 0, \ldots, 0) = f_0(x)$ for any $x \in \Re$. By Theorem 3.4, $f = f_1 \circ h_1^{-1}$ is almost continuous. Suppose that $f \cdot g$ is almost continuous. Then $(f \cdot g)|h(\Re)$ is almost continuous (by Theorem 2.1), and therefore, $(f \cdot g) \circ h \in \mathcal{A}(\Re, \Re)$. But $(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h) = f_0 \cdot (g \circ h)$, a contradiction.

(2.1) For $n = 1$ see (1.3). Assume that $g = (g_1, \ldots, g_n) \in \mathcal{M}_m(\Re^k, Z^n)$. We shall verify that $g \in \mathcal{M}(\Re^k, Z^n)$. Let $h : I \longrightarrow \Re^k$ be a homeomorphic injection such that $g \circ h$ is discontinuous at 0. We can assume that $g_1 \circ h$ is discontinuous at 0. Let $h(0) = x_0$. From (1.2) it follows that $g(x_0) = 0$. Note that $\sum_{i=1}^n g_i \in \mathcal{M}_m(\mathcal{A}(\Re^k, Z))$. Indeed, this follows from the fact that $f_0 = (f, \ldots, f)$ is almost continuous for any $f \in \mathcal{A}(\Re, [0, \infty))$ (as the composition of f and continuous function d from $\Re$ into $\Re^n$ defined by $d(x) = (x, \ldots, x)$), and $(\sum_{i=1}^n g_i) \cdot f = g \cdot f_0 \in \mathcal{A}(\Re^k, Z)$. Hence $(\sum_{i=1}^n g_i) \cdot h$ is almost continuous whenever so is h. Observe that the function $(\sum_{i=1}^n g_i) \circ h$ is discontinuous at 0. Since $(\sum_{i=1}^n g_i) \circ h \in \mathcal{M}_m(\mathcal{A}(I, Z))$, there is a sequence $(x_j)_j$ converging to 0 such that $(\sum_{i=1}^n g_i)(h(x_j)) = 0$ for each j. Since $g_i \geq 0$ for each $i \leq n$, $g_i(h(x_j)) = 0$ for all $j \in N$ and $i \leq n$. Hence $g \circ h \in \mathcal{M}(I, Z^n)$.

(2.2) The inclusion $\mathcal{M}_m(\mathcal{A}(\Re, Z^n)) \subset \mathcal{M}(\Re, Z^n)$ follows from the condition (2.1). Now assume that $g \in \mathcal{M}(\Re, [0, \infty)^n)$. Then for arbitrary

$f \in \mathcal{A}(\Re, Z^n)$ the product $f \cdot g$ satisfies all assumptions of Lemma 4.1, so it is almost continuous. Therefore $g \in \mathcal{M}_m(\mathcal{A}(\Re, Z^n))$.

(3) The inclusion $\mathcal{M}_m(\mathcal{A}(\Re^k, Z^n)) \supset \mathcal{C}(\Re^k, Z^n)$ follows from Theorems 4.4 and 3.3. Now suppose that $f = (f_1, \dots, f_m) \in \mathcal{M}_m(\mathcal{A}(\Re^k, Z^n))$. Fix $i \leq n$ and observe that $f_0 = ln \circ f_i \in \mathcal{M}_a(\mathcal{A}(\Re^k, \Re))$. Indeed, if $g \in \mathcal{A}(\Re^k, \Re)$ then $g_0 = exp \circ g \in \mathcal{A}(\Re^k, Z)$ and consequently $h = (h_1, \dots, h_m)$, where $h_i = g_0$ and $h_j \equiv 0$ for $j \neq i$, is almost continuous. Thus $f_i \cdot g_0 = f \cdot h$ is almost continuous and therefore $f_0 + g = ln(f_i \cdot g_0)$ is almost continuous, too. Hence $f_0 \in \mathcal{M}_a(\mathcal{A}(\Re^k, \Re))$ and, by Theorem 6.3, it is continuous and so is $f_i = exp \circ f_0$. Thus f is continuous.

Q.E.D.

Lemma 6.2 *Let F be a compact subset of a metric space X, $f \in \mathcal{A}(X, \Re^k)$ and $f|F$ be continuous. Then each open neighbourhood G of f in $X \times \Re^k$ includes a continuous function $g : X \longrightarrow \Re^k$ such that $g|F = f|F$.*

Proof. First suppose that $f|F \equiv 0$ and G is a neighbourhood of f. Since $F \times \{0\}$ is compact, there exists a positive ε such that $B_X(x, \varepsilon) \times B_{\Re^k}(0, \varepsilon) \subset G$ for all $x \in F$. Since $f \subset G_1 = G \setminus (F \times (\Re^k \setminus B_{\Re^k}(0, \varepsilon)))$, there exists a continuous function $h : X \longrightarrow \Re^k$ contained in G_1. For every $x \in F$ choose δ_x such that $0 < \delta_x < \varepsilon/2$ and $\|h(z)\| < \varepsilon$ for $z \in B_X(x, \delta_x)$. Let δ be Lebesgue number of the covering $\{B_X(x, \delta_x) : x \in F\}$ of F and let $A = \bigcup_{x \in F} B_X(x, \delta)$. Then $\|h(z)\| < \varepsilon$ for $z \in A$, $A \times B_{\Re^k}(0, \varepsilon) \subset G$ and the function $g(z) = min(\delta, dist(z, F)) \cdot h(z)/\delta$ is continuous, $g \subset G$ and $g(x) = 0$ for $x \in F$.

Now we consider an arbitrary $f \in \mathcal{A}(X, \Re^k)$ such that $f|F$ is continuous. Let $G \subset X \times \Re^k$ be a neighbourhood of f and let f^* be a continuous extension of $f|F$ onto whole X. Then the function $h : X \times \Re^k \longrightarrow X \times \Re^k$ defined by $h(x, y) = (x, y - f^*(x))$ is a homeomorphism. Therefore $G_1 = h(G)$ is an open neighbourhood of an almost continuous function $f_1 = f - f^*$ and, moreover, $f_1|F \equiv 0$. Thus there exists a continuous function $g_1 : X \longrightarrow \Re^k$ such that $g_1 \subset G_1$ and $g_1|F \equiv 0$. Then $g = h^{-1} \circ g_1 = g_1 + f^*$ is a continuous function contained in G and $g|F = f|F$.

Q.E.D.

Lemma 6.3 *Suppose that X is a locally compact metric space, F is a compact subset of X and $f : X \longrightarrow \Re^k$ satisfies the following conditions:*

(1) $f|F \equiv 0$,

(2) $f|\overline{U}$ *is almost continuous for every component* U *of the set* $X \setminus F$.

Then f *is almost continuous.*

Proof. Let G be an open neighbourhood of f and U be a component of $X \setminus F$. Then $f|\overline{U}$ is almost continuous, $f|fr(U) \equiv 0$ and $fr(U)$ is compact. By Lemma 6.2 there exists a continuous function $g_U : \overline{U} \longrightarrow \Re^k$ such that $g_U \subset G$ and $g_U|fr(U) \equiv 0$. Since $F \times \{0\}$ is compact, there exists a positive ε such that $V \times \{0\} \subset G$, where $V = \{x \in X : dist(x, F) < \varepsilon\}$. Since X is locally compact, there exists an open set W such that $F \subset W \subset \overline{W} \subset V$ and $\overline{W}$ is compact (cf. [19], Theorem 2, p. 193). Then $E = \overline{W} \setminus W$ is compact and $E \subset X \setminus F$. Let $\{U_1, \ldots, U_n\}$ be a finite subcovering of E chosen from the family of all components of $X \setminus F$. Note that for each component U of $X \setminus F$ one of the following cases holds: $U \subset X \setminus \overline{W}$ or $U = U_i$ for some $i \leq n$ or $U \subset W$. Hence the function $g : X \longrightarrow \Re^k$ given by

$$g(x) = \begin{cases} g_U(x) & \text{if } x \in U \subset X \setminus \overline{W} \\ g_{U_i}(x) & \text{if } x \in U_i,\ 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}$$

is continuous. Clearly, $g \subset G$.

Q.E.D.

For any topological space X and $Y \subset \Re^k$ we shall denote by $\mathcal{M}^*(X, Y)$ the family of all functions $f : X \longrightarrow Y$ such that $[f = 0]$ is compact and $f|\overline{U}$ is continuous for each component of U of the set $[f \neq 0]$.

Theorem 6.8

(1) $\mathcal{M}^*(X, \Re^m) \subset \mathcal{A}(X, \Re^m) \cap \mathcal{M}(X, \Re^m)$ *for each locally compact metric space* X.

(2) $\mathcal{M}^*(I, \Re^m) = \mathcal{M}(I, \Re^m)$.

(3) $\mathcal{A}(I^2, \Re) \cap \mathcal{M}(I^2, \Re) \setminus \mathcal{M}^*(I^2, \Re) \neq \emptyset$.

(4) $\mathcal{M}(I^2, \Re) \setminus \mathcal{A}(I^2, \Re) \neq \emptyset$.

P r o o f . The inclusions $\mathcal{M}^*(X,\Re^m) \subset \mathcal{M}(X,\Re^m)$ (for any X) and $\mathcal{M}(I,\Re^m) \subset \mathcal{M}^*(I,\Re^m)$ are easy to observe. By Lemma 6.3, $\mathcal{M}^*(X,\Re^m) \subset \mathcal{A}(X,\Re^m)$ for any locally compact metric space X.

(3) For $n \in N$ put $J_n = \{1/n\} \times I$ and define the continuous function $f_n : J_n \longrightarrow I$ such that:

(i) if n is even then $f_n|J_n \equiv 1$,

(ii) if $n \equiv 1 \pmod 4$ then $[f_n = 0] = \{1/n\} \times [0, 1-1/n]$ and $rng(f_n) = [0, 1/n]$,

(iii) if $n \equiv 3 \pmod 4$ then $[f_n = 0] = \{1/n\} \times [1/n, 1]$ and $rng(f_n) = [0, 1/n]$.

Moreover let $f_0 : \{0\} \times I \longrightarrow I$ be the function defined by $f_0 \equiv 0$. Let $g : (0,1] \times I \longrightarrow I$ be a continuous extension of the function $\bigcup_{n=1}^{\infty} f_n$ such that $[g = 0] = \bigcup_{n=1}^{\infty}[f_n = 0]$ and let $f = f_0 \cup g$. Then $f \in \mathcal{A}(I^2, I) \cap \mathcal{M}(I^2, I) \setminus \mathcal{M}^*(I^2, I)$.

(4) Let $f_0 : I \times [-2,2] \longrightarrow \Re$ be defined by:

$$f_0(x,y) = \begin{cases} 1 - |y - sin(1/x)| & \text{if } x > 0 \text{ and } |y - sin(1/x)| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Obviously $f_0 \in \mathcal{M}(I\times[-2,2],\Re)$. Suppose that $f_0 \in \mathcal{A}(I\times[-2,2],\Re)$. Then $A = \{(x,y) : \ x = 0 \text{ or } (x > 0 \text{ and } y = sin(1/x))\}$ is a continuum, $f_0|A$ is almost continuous and $rng(f_0|A) = \{0,1\}$, contrary to Theorem 1.7. Thus $f_0 \in \mathcal{M}(I \times [-2,2],\Re) \setminus \mathcal{A}(I \times [-2,2],\Re)$. Now let $h : I^2 \longrightarrow I \times [-2,2]$ be a homeomorphism and $f = f_0 \circ h$. Then $f \in \mathcal{M}(I^2,\Re) \setminus \mathcal{A}(I^2,\Re)$.

Q.E.D.

Theorem 6.9

(1) $\mathcal{M}^*(X,\Re) \subset \mathcal{M}_m(X,\Re)$ *for any locally connected metric space* X.

(2) $\mathcal{M}^*(I^k,\Re) \subset \mathcal{M}_m(I^k,\Re) \subset \mathcal{M}(I^k,\Re)$.

P r o o f . (1) Assume that $f \in \mathcal{M}^*(X,\Re)$, $g \in \mathcal{A}(X,\Re)$ and put $F = [f = 0]$. Then $F \subset [f \cdot g = 0]$ and, by Theorems 4.4 and 3.3, $(f \cdot g)|\overline{U}$ is almost continuous for each component U of the set $X \setminus F$. By Lemma 6.3 $f \cdot g$ is almost continuous.

(2) We need only to prove the second inclusion. Suppose that $g \in \mathcal{M}_m(\mathcal{A}(I^k, \Re)) \setminus \mathcal{M}(I^k, \Re)$ and $h : I \longrightarrow I^k$ is a homeomorphic injection such that $g \circ h \notin \mathcal{M}(I, \Re)$. Let $h_1 : I^k \longrightarrow h(I)$ be a retraction. Since $g \circ h \notin \mathcal{M}(I, \Re)$, there exists $f_0 \in \mathcal{A}(I, \Re)$ such that $f_0 \cdot (g \circ h) \notin \mathcal{A}(I, \Re)$. Then $f_1 = f_0 \circ h^{-1} \circ h_1 \in \mathcal{A}(I^k, \Re)$ and therefore $f_1 \cdot g \in \mathcal{A}(I^k, \Re)$. Hence $(f_1 \cdot g)|h(I) \in \mathcal{A}(h(I), \Re)$ and $(f_1 \cdot g) \circ h \in \mathcal{A}(I, \Re)$, but $(f_1 \cdot g) \circ h = (f_1 \circ h) \cdot (g \circ h) = f_0 \cdot (g \circ h) \notin \mathcal{A}(I, \Re)$, a contradiction.

Q.E.D.

Problem 6.3 *Characterize classes $\mathcal{M}_m(\mathcal{A}(\Re^k and \Re^n))$, $\mathcal{M}_m(\mathcal{A}(I^k, \Re^n))$ for positive integers k, n.*

6.3 Maxima and minima.

Suppose that Y is a lattice. If $\mathcal{F}$ is a family of functions from X into Y then the symbol $\mathcal{L}(\mathcal{F})$ denotes the lattice generated by $\mathcal{F}$, i.e. the smallest lattice of functions containing $\mathcal{F}$.

Proposition 6.4 *Suppose that (X, Y) is a (K, G) pair with infinite blocking family $\mathcal{K}$ and Y is a lattice. Then $\mathcal{L}(\mathcal{A}(X, Y)) = Y^X$.*

More precisely, any function f from X into Y can be expressed as

$$min(max(f_1, f_2), max(f_3, f_4)),$$

where f_1, f_2, f_3, f_4 are almost continuous.

P r o o f . Assume that $card(\mathcal{K}) = \lambda$ and $\{X_1, X_2, X_3, X_4\}$ is a partition of X such that $card(X_i \cap K) \geq \lambda$ for each $K \in \mathcal{K}$ and $i = 1, 2, 3, 4$ (such a partition exists by Lemma 5.1). Fix $f : X \longrightarrow Y$ and $i \in \{1, 2, 3, 4\}$. For each $\alpha < \lambda$ choose $(x_{i,\alpha}, y_{i,\alpha}) \in K_\alpha$ such that $x_{i,\alpha} \in X_i$ and $x_{i,\alpha} \neq x_{i,\beta}$ for $\alpha \neq \beta$ and $\beta < \lambda$. Now we define the function f_i by $f_i(x_{i,\alpha}) = y_{i,\alpha}$ for $\alpha < \lambda$ and $f_i(x) = f(x)$ for other x. One can easily verify that all f_i are almost continuous and $f = min(max(f_1, f_2), max(f_3, f_4))$.

Q.E.D.

Remark 6.2 *If f_1, f_2, f_3 are defined as above, then $f = max(h_1, h_2)$, where $h_1 = min(max(f_1, f_2), f_3)$ and $h_1 = min(max(f_1, f_3), f_2)$.*

Corollary 6.7 *Each function $f : \Re \longrightarrow \Re$ can be expressed as*

$$min(max(f_1, f_2), max(f_3, f_4)),$$

where f_1, f_2, f_3, f_4 are almost continuous [47].

Remark 6.3 *If $f : \Re \longrightarrow \Re$ is measurable (has the Baire property), then the functions f_1, f_2, f_3, f_4 from Corollary 6.7 may be chosen measurable (with the Baire property).*

For arbitrary topological space X and lattice Y we shall denote by $\ell(X, Y)$ the order of the lattice $\mathcal{L}(\mathcal{A}(X, Y))$, i.e. the least positive integer k such that for any $f \in \mathcal{L}(\mathcal{A}(X, Y))$ there exists a subset $\mathcal{F}_0 \subset \mathcal{A}(X, Y)$ such that $card(\mathcal{F}_0) = k$ and $f \in \mathcal{L}(\mathcal{F}_0)$.

Corollary 6.8 $\ell(\Re, \Re) = 3$.

Proof. By Remark 6.2, $\ell(X, Y) \leq 3$ for any (K, G) pair with an infinite blocking family. On the other hand, the function $f : \Re \longrightarrow \Re$ defined by $f(x) = x$ for $x \in \{-1, 1\}$ and $f(x) = 0$ for $x \notin \{-1, 1\}$ cannot be expressed as the minimum or the maximum of two Darboux functions, so $\ell(\Re, \Re) > 2$. Q.E.D.

Proposition 6.5 *Suppose that (X, Y) is a (K, G) pair with an infinite blocking family $\mathcal{K}$ and $\leq$ is a partial order in Y. If a function $f : X \longrightarrow Y$ satisfies the condition:*

$(*)$ *$card(\{x \in X : f(x) \geq y$ for some $y \in K_x\}) \geq card(\mathcal{K})$ for every $K \in \mathcal{K}$,*

then f can be represented as a maximum of two almost continuous functions.

Proof. Let $card(\mathcal{K}) = \lambda$. Note that the condition $(*)$ implies the existence of two disjoint subsets A, B of X such that $card(\{x \in A : (x, y) \in K$ and $f(x) \geq y$ for some $y \in Y\}) \geq \lambda$ and $card(\{x \in B : (x, y) \in K$ and $f(x) \geq y$ for some $y \in Y\}) \geq \lambda$ for each $K \in \mathcal{K}$. Let $(K_\alpha)_{\alpha<\lambda}$ be a sequence of all blocking sets from the family $\mathcal{K}$. For every $\alpha < \lambda$ choose points $(a_\alpha, a'_\alpha), (b_\alpha, b'_\alpha) \in K_\alpha$ such that:

(1) $a_\alpha \in A \setminus \{a_\beta : \beta < \alpha\}$ and $f(a_\alpha) \geq a'_\alpha$,

(2) $b_\alpha \in B \setminus \{b_\beta : \beta < \alpha\}$ and $f(b_\alpha) \geq b'_\alpha$.

Define f_1, f_2 in the following way: $f_1(a_\alpha) = a'_\alpha$ for $\alpha < \lambda$ and $f_1(x) = f(x)$ for other x. Similarly, $f_2(b_\alpha) = b'_\alpha$ for $\alpha < \lambda$ and $f_2(x) = f(x)$ otherwise. Since f_1, f_2 meet all blocking sets from the family $\mathcal{K}$, they are almost continuous. Moreover, $f = max(f_1, f_2)$.

Q.E.D.

Theorem 6.10 *Each function $f : I \longrightarrow \Re$ satisfying the following condition*

(♣) $\quad [f \geq n]$ *is c-dense in I for any positive integer n*

can be represented as a maximum of two almost continuous functions f_1, f_2. Moreover, if f is measurable or has the Baire property, then f_1, f_2 may be chosen measurable or with the Baire property as well.

P r o o f . Suppose that $f : I \longrightarrow \Re$ satisfies the condition (♣). Let $\mathcal{K}$ be the family of all minimal blocking sets in $I \times \Re$. It is sufficient to verify that the condition $(*)$ from Proposition 6.5 is satisfied. Fix $K \in \mathcal{K}$. Since $dom(K) = \bigcup_{n=1}^{\infty} K_n$, where $K_n = dom(K \cap (I \times [-n, n]))$, K_{n_0} is of the second category for some positive integer n_0. Since K_{n_0} is closed, it has non-empty interior. Let J be a non-empty open interval contained in K_{n_0}. By (♣), $card(\{x \in J : f(x) \geq n_0\}) = 2^\omega$. Since for each $x \in J$ there exists $y \in [-n_0, n_0]$ such that $(x, y) \in K$, $J \subset \{x \in I : (x, y) \in K$ and $f(x) \geq y$ for some $y \in \Re\}$ and therefore, $(*)$ holds.

Finally, remark that if f is measurable (has the Baire property), then we can choose disjoint sets of measure zero (of the first category) A, B such that for any real r the sets $A \cap [f \geq r]$ and $B \cap [f \geq r]$ are c-dense in I. Now we can choose elements a_α, b_α (as in the proof of Proposition 6.5) from such sets A and B. Then f_1, f_2 will be measurable (have the Baire property).

Q.E.D.

Corollary 6.9 *Every $f \in \mathcal{D}^*$ can be represented as a maximum of two almost continuous functions.*

For arbitrary function $f : \Re \longrightarrow \Re$ and $x \in \Re$ let $K_c^+(f, x)$ denote the right hand c-cluster set of f at x, i.e. $K_c^+(f, x) = \bigcap\{C^+(f|\Re \setminus B, x) : card(B) < 2^\omega\}$. Similarly we define the left hand c-cluster set of f at x (denoted by $K_c^-(f, x)$). It is known that a function $f : \Re \longrightarrow \Re$ is a maximum of two Darboux functions iff it satisfies the following condition:

(♠) $\quad f(x) \leq min(max(K_c^+(f, x)), max(K_c^-(f, x)))$ for each $x \in \Re$ [12].

Problem 6.4 *Is every function $f : \Re \longrightarrow \Re$ satisfying (♠) a maximum of two almost continuous functions ?*

Let X be a topological space and $\mathcal{X}, \mathcal{Y}$ be arbitrary families of real functions defined on X. We define the following classes of functions:

$$\mathcal{M}_{max}(\mathcal{X},\mathcal{Y}) = \{f \in \mathcal{X} : max(f,g) \in \mathcal{Y} \text{ for all } g \in \mathcal{X}\},$$

$$\mathcal{M}_{min}(\mathcal{X},\mathcal{Y}) = \{f \in \mathcal{X} : min(f,g) \in \mathcal{Y} \text{ for all } g \in \mathcal{X}\},$$

$$\mathcal{M}_{l}(\mathcal{X},\mathcal{Y}) = \{f \in \mathcal{X} : max(f,g), min(f,g) \in \mathcal{Y} \text{ for all } g \in \mathcal{X}\}.$$

Clearly, $\mathcal{M}_l(\mathcal{X},\mathcal{Y}) = \mathcal{M}_{max}(\mathcal{X},\mathcal{Y}) \cap \mathcal{M}_{min}(\mathcal{X},\mathcal{Y})$. We shall write $\mathcal{M}_{max}(\mathcal{X})$, $\mathcal{M}_{min}(\mathcal{X})$ and $\mathcal{M}_l(\mathcal{X})$ instead of $\mathcal{M}_{max}(\mathcal{X},\mathcal{X})$, $\mathcal{M}_{min}(\mathcal{X},\mathcal{X})$ and $\mathcal{M}_l(\mathcal{X},\mathcal{X})$, respectively. The last family is called the *maximal lattice class* for $\mathcal{X}$.

Theorem 6.11 *If $\mathcal{X} \in \{\mathcal{A}(\Re,\Re),\ Conn(\Re,\Re),\ \mathcal{D}(\Re,\Re)\}$ then*

(1) $\mathcal{C}(\Re,\Re) \subset \mathcal{M}_{max}(\mathcal{A}(\Re,\Re),\mathcal{X}) \subset \mathcal{D}usc(\Re,\Re)$,

(2) $\mathcal{C}(\Re,\Re) \subset \mathcal{M}_{min}(\mathcal{A}(\Re,\Re),\mathcal{X}) \subset \mathcal{D}lsc(\Re,\Re)$,

(3) $\mathcal{M}_l(\mathcal{A}(\Re,\Re),\mathcal{X}) = \mathcal{C}(\Re,\Re)$

P r o o f . Those relations are proved in [30] for $\mathcal{X} = \mathcal{A}(\Re,\Re)$. The proof is analogous for other $\mathcal{X}$.

Q.E.D.

Arguments similar to those used in the proofs of Theorem 6.3 and Corollary 6.3 imply the following theorem.

Theorem 6.12 *The equality $\mathcal{M}_l(\mathcal{A}(\Re^k,\Re^m)) = \mathcal{C}(\Re^k,\Re^m)$ holds for all positive integers k, m.*

Problem 6.5 *Describe the classes $\mathcal{M}_{max}(\mathcal{A}(\Re^k,\Re^m))$ and $\mathcal{M}_{min}(\mathcal{A}(\Re^k,\Re^m))$ for positive integers k, m.*

7 Insertions of functions.

Example 7.1 *There exist almost continuous, measurable functions $f, g : \Re \longrightarrow \Re$ with the Baire property such that $f < g$ and f, g admit no Darboux function between them.*

Indeed, let $(K_\alpha)_{\alpha<2^\omega}$ be the sequence of all blocking sets in $\Re \times \Re$. Let Z_0, Z_1, Z_2 be pairwise disjoint, c-dense subsets of $\Re$ of measure zero and of the first category. Choose sequences $(x_{i,\alpha}, y_{i,\alpha})_{\alpha<2^\omega}$ for $i = 1,2$ such that $(x_{i,\alpha}, y_{i,\alpha}) \in K_\alpha$ and $x_{i,\alpha} \in Z_i \setminus \{x_{i,\beta} : \beta < \alpha\}$ for $i = 1,2$ and $\alpha < 2^\omega$. Define the functions f, g in the following way:

$$f(x) = \begin{cases} y_{1,\alpha} & \text{if } x = x_{1,\alpha},\ \alpha < 2^\omega \\ y_{2,\alpha} - 1 & \text{if } x = x_{2,\alpha},\ y_{2,\alpha} \le 0,\ \alpha < 2^\omega \\ y_{2,\alpha}/2 & \text{if } x = x_{2,\alpha},\ y_{2,\alpha} > 0,\ \alpha < 2^\omega \\ -2 & \text{if } x \in Z_0 \\ 1 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} y_{2,\alpha} & \text{if } x = x_{2,\alpha},\ \alpha < 2^\omega \\ y_{1,\alpha} + 1 & \text{if } x = x_{1,\alpha},\ y_{1,\alpha} \ge 0,\ \alpha < 2^\omega \\ y_{1,\alpha}/2 & \text{if } x = x_{1,\alpha},\ y_{1,\alpha} < 0,\ \alpha < 2^\omega \\ -1 & \text{if } x \in Z_0 \\ 2 & \text{otherwise} \end{cases}$$

Then f and g are almost continuous, $f < g$ and there is no Darboux function h between them (cf. [12]). Indeed, if h is a function such that $f < h < g$ then $h(x) < 0$ for $x \in Z_0$, $h(x) > 0$ for $x \in \Re \setminus (Z_0 \cup Z_1 \cup Z_2)$ and $h(x) \neq 0$ for all $x \in \Re$.

Theorem 7.1 *Assume that $f, g \in \mathcal{A}(X, \Re)$, $f < g$ and at least one of f, g is continuous. Then there exists an almost continuous h between f and g.*

Proof. Obviously the function $h = (f+g)/2$ has the required property.

Proposition 7.1 *Assume that (X, Y) is a (K, G) pair with infinite blocking family $\mathcal{K}$ and $(Y, \le)$ is a partially ordered set. If $\mathcal{F}$ is a family of functions from X into Y satisfying the following conditions:*

(1) functions from $\mathcal{F}$ are commonly bounded, i.e.

$$\forall x \in X \quad \exists l(x) \quad \exists u(x) \quad \forall f \in \mathcal{F} \quad l(x) \le f(x) \le u(x),$$

(2) for each $K \in \mathcal{K}$ we have:

- $card(\{x \in X : \exists y \in Y \ \forall f \in \mathcal{F} \ (x,y) \in K \text{ and } f(x) \geq y\}) \geq card(\mathcal{K})$,

and

- $card(\{x \in X : \exists y \in Y \ \forall f \in \mathcal{F} \ (x,y) \in K \text{ and } f(x) \leq y\}) \geq card(\mathcal{K})$,

then there exist almost continuous functions $g_l, g_u : X \longrightarrow Y$ such that $g_l \leq f \leq g_u$ for all $f \in \mathcal{F}$.

P r o o f . Let $card(\mathcal{K}) = \lambda$. Let $(K_\alpha)_{\alpha<\lambda}$ be a sequence of all sets from $\mathcal{K}$. By (2) we can choose disjoint sets $A_1, A_2 \subset X$ such that $card(\{x \in A_1 : \exists y \in Y \ \forall f \in \mathcal{F} \ (x,y) \in K$ and $f(x) \geq y\}) \geq \lambda$ and $card(\{x \in A_2 : \exists y \in Y \ \forall f \in \mathcal{F} \ (x,y) \in K$ and $f(x) \leq y\}) \geq \lambda$ for each $K \in \mathcal{K}$. Let $(a_\alpha, a'_\alpha)_{\alpha<\lambda}$, $(b_\alpha, b'_\alpha)_{\alpha<\lambda}$ be sequences of points such that $(a_\alpha, a'_\alpha), (b_\alpha, b'_\alpha) \in K_\alpha$, $a'_\alpha \leq f(x)$, $f(b_\alpha) \leq b'_\alpha$ for each $f \in \mathcal{F}$ and $\alpha < \lambda$, and moreover, $a_\alpha \neq a_\beta$, $b_\alpha \neq b_\beta$ whenever $\alpha \neq \beta$. Then the functions g_l, g_u defined by $g_l(a_\alpha) = a'_\alpha$, $g_u(b_\alpha) = b'_\alpha$ for $\alpha < \lambda$ and $g_l(x) = l(x), g_u(x) = u(x)$ for other x, have the required properties.

Q.E.D.

Theorem 7.2 *For each function $f : I \longrightarrow \Re$ for which $\{-\infty, \infty\} \subset K_c(f,x)$ for each $x \in I$ there exist almost continuous functions g, h such that $g < h$ and $f = (g+h)/2$ (hence $g < f < h$). Moreover, if f is measurable (has the Baire property), then g and h can be taken measurable (with the Baire property).*

P r o o f . Let $(F_\alpha)_{\alpha<2^\omega}$ be the sequence of all minimal blocking sets in $I \times \Re$. For each ordinal $\alpha < 2^\omega$ there exist a positive integer n_α and a non-degenerate interval J_α such that $J_\alpha \subset dom(F_\alpha \cap (I \times [-n_\alpha, n_\alpha]))$. For every $\alpha < 2^\omega$ choose subsets $A_\alpha \subset J_\alpha \cap [f < -n_\alpha]$, $B_\alpha \subset J_\alpha \cap [f > n_\alpha]$ such that $card(A_\alpha) = card(B_\alpha) = 2^\omega$. Note that $A_\alpha \cap B_\beta = \emptyset$ for $\alpha, \beta < 2^\omega$. Let $(a_\alpha, a'_\alpha)_{\alpha<2^\omega}$, $(b_\alpha, b'_\alpha)_{\alpha<2^\omega}$ be sequences of points such that $(a_\alpha, a'_\alpha), (b_\alpha, b'_\alpha) \in F_\alpha \cap (I \times [-n_\alpha, n_\alpha])$, $a_\alpha \in A_\alpha \setminus \{a_\beta : \beta < \alpha\}$ and $b_\alpha \in B_\alpha \setminus \{b_\beta : \beta < \alpha\}$ for any $\alpha < 2^\omega$. Now define the functions g and h in the following way:

$$h(x) = \begin{cases} a'_\alpha & \text{for } x = a_\alpha,\ \alpha < 2^\omega \\ 2f(b_\alpha) - b'_\alpha & \text{for } x = b_\alpha,\ \alpha < 2^\omega \\ f(x) + 1 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} b'_\alpha & \text{for } x = b_\alpha,\ \alpha < 2^\omega \\ 2f(a_\alpha) - a'_\alpha & \text{for } x = a_\alpha,\ \alpha < 2^\omega \\ f(x) - 1 & \text{otherwise} \end{cases}$$

Clearly, g, h are almost continuous and $f = (g + h)/2$.

Finally observe that if f is measurable (has the Baire property), then sets $[f > n]$ and $[f < -n]$ are measurable (have the Baire property) for every positive integer n and we can choose c-dense in I sets of measure zero and of the first category $A_n \subset [f > n]$ and $B_n \subset [f < -n]$. Sets $A = \bigcup_{n=1}^\infty A_n$, $B = \bigcup_{n=1}^\infty B_n$ have measure zero (are of the first category) and we continue as in the proof of general case with $a_\alpha \in A$, $b_\alpha \in B$ for $\alpha < 2^\omega$. Since $[g \neq f] \cup [h \neq f] \subset A \cup B$, g and h are measurable (have the Baire property).

Q.E.D.

Corollary 7.1 *For every function f from $\mathcal{D}^*(\Re, \Re)$ there exist almost continuous functions g and h such that $g < f < h$.*

8 Stationary and determining sets.

Let $\mathcal{F}$ be a family of functions defined on X into Y. A subset E of X is called *stationary* for $\mathcal{F}$ provided that each member of $\mathcal{F}$ which is constant on E must be constant on all of X. We shall denote by $\boldsymbol{S}(\mathcal{F})$ the collection of all stationary sets for the class $\mathcal{F}$. A set E is called a *determining set* for $\mathcal{F}$ provided that each two functions from $\mathcal{F}$ which coincide on E must coincide on whole X. The class of all determining sets for $\mathcal{F}$ will be denoted by $\boldsymbol{D}(\mathcal{F})$. A set $E \subset X$ is called a *restrictive set* for the pair $(\mathcal{F}_1, \mathcal{F}_2)$ of families of functions from X into Y provided that $f_1 = f_2$ whenever $f_1 \in \mathcal{F}_1$, $f_2 \in \mathcal{F}_2$ and $f_1|E = f_2|E$. The class of all restrictive sets for $(\mathcal{F}_1, \mathcal{F}_2)$ will be denoted by $\boldsymbol{R}(\mathcal{F}_1, \mathcal{F}_2)$ [9]. Note that

(1) if $Const(X, Y) \subset \mathcal{F}$ then $\boldsymbol{D}(\mathcal{F}) \subset \boldsymbol{S}(\mathcal{F})$,

(2) $\boldsymbol{R}(\mathcal{F},\mathcal{F}) = \boldsymbol{D}(\mathcal{F})$ and $\boldsymbol{R}(Const,\mathcal{F}) = \boldsymbol{S}(\mathcal{F})$

(3) if $\mathcal{F}_1 \subset \mathcal{F}_2$ then $\boldsymbol{S}(\mathcal{F}_2) \subset \boldsymbol{S}(\mathcal{F}_1)$ and $\boldsymbol{D}(\mathcal{F}_2) \subset \boldsymbol{D}(\mathcal{F}_1)$,

(4) if $\mathcal{F}_1 \subset \mathcal{F}_2$ then $\boldsymbol{R}(\mathcal{F}_2,\mathcal{F}) \subset \boldsymbol{R}(\mathcal{F}_1,\mathcal{F})$ for every family $\mathcal{F}$ of functions from X into Y.

Theorem 8.1 *A necessary and sufficient condition for $E \subset I$ to be a stationary set for the class $\mathcal{A}(I,\Re^k)$ is that $card(I \setminus E) < 2^\omega$.*

P r o o f . Since $\mathcal{A} \subset \mathcal{D}$, $\boldsymbol{S}(\mathcal{D}(I,\Re)) \subset \boldsymbol{S}(\mathcal{A}(I,\Re))$. It is known [2] (and easy to obtain, see e.g. [9], p. 200) that $E \in \boldsymbol{S}(\mathcal{D}(I,\Re))$ iff $card(I \setminus E) < 2^\omega$. Thus $card(I \setminus E) < 2^\omega$ implies $E \in \boldsymbol{S}(\mathcal{A}(I,\Re)) \subset \boldsymbol{S}(\mathcal{A}(I,\Re^k))$.

Now assume that $K = I \setminus E$ and $card(K) = 2^\omega$. Let K_0 be the set of all points of bilateral c-condensation of K. Obviously K_0 is non-empty and bilaterally c-dense in itself. Arrange all minimal blocking sets in $I \times \Re^k$ such that $dom(F) \cap K_0 \neq \emptyset$ in a sequence $(F_\alpha)_{\alpha<2^\omega}$. Note that $card(dom(F_\alpha \cap K_0)) = 2^\omega$ for $\alpha < 2^\omega$. Fix arbitrary $z \in K_0$ and choose a sequence of points $(x_\alpha, y_\alpha)_{\alpha<2^\omega}$ such that $(x_\alpha, y_\alpha) \in F_\alpha$, $x_\alpha \neq z$ for all $\alpha < 2^\omega$ and $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Let $f : I \longrightarrow \Re^k$ be the function defined by $f(z) = (1,\ldots,1)$, $f(x_\alpha) = y_\alpha$ for $\alpha < 2^\omega$ and $f(x) = 0$ for other x. Observe that f intersects each minimal blocking set F in $I \times \Re^k$. Indeed, if $F = F_\alpha$ for some $\alpha < 2^\omega$ then $(x_\alpha, y_\alpha) \in f \cap F$. In the other case $dom(F) \subset \overline{J}$, where J is a component of the set $I \setminus \overline{K_0}$. Since $rng(F) = \Re^k$, $(x,0) \in f \cap F$ for some $x \in dom(F)$. Thus f is almost continuous, $f|E \equiv 0$ but $f \not\equiv 0$, therefore E is not stationary for $\mathcal{A}(I,\Re^k)$.

Q.E.D.

Corollary 8.1 $E \in \boldsymbol{S}(\mathcal{A}(\Re,\Re^k))$ *iff* $card(\Re \setminus E) < 2^\omega$.

Corollary 8.2 *Since* $\mathcal{A}(\Re,\Re) \subset Conn(\Re,\Re) \subset \mathcal{D}(\Re,\Re)$, *and* $\boldsymbol{S}(\mathcal{A}(\Re,\Re)) = \boldsymbol{S}(\mathcal{D}(\Re,\Re))$, $E \in \boldsymbol{S}(Conn(\Re,\Re))$ *iff* $card(\Re \setminus E) < 2^\omega$.

Theorem 8.2 *The only determining set for the classes* $\mathcal{A}(\Re,\Re^k)$, $Conn(\Re,\Re^k)$ *is* $\Re$.

Proof. For $k = 1$ this follows from the inclusions $\mathcal{DB}_1 \subset \mathcal{A}(\Re,\Re) \subset Conn(\Re,\Re) \subset \mathcal{D}(\Re,\Re)$, the condition (3) before Theorem 8.1 and the equalities $\boldsymbol{D}(\mathcal{DB}_1) = \boldsymbol{D}(\mathcal{D}) = \{\Re\}$ [14]. For $k > 1$ this is a consequence of the inclusions $\boldsymbol{D}(\mathcal{A}(\Re,\Re^k)) \subset \boldsymbol{D}(\mathcal{A}(\Re,\Re))$ and $\boldsymbol{D}(Conn(\Re,\Re^k)) \subset \boldsymbol{D}(Conn(\Re,\Re))$.

Q.E.D.

The following equalities are easy consequences of Theorems 8.1 and 8.2 and the conditions before Theorem 8.1 (cf. [9], Theorem 2.1, p. 207).

Corollary 8.3 *In the class of real functions defined on $\Re$ the following equalities hold:*

(1) $E \in \boldsymbol{R}(\mathcal{C},\mathcal{X})$ iff $card(\Re \setminus E) < 2^\omega$, for $\mathcal{X} \in \{\mathcal{A}, Conn, \mathcal{D}\}$,

(2) $\boldsymbol{R}(Conn,\mathcal{D}) = \boldsymbol{R}(\mathcal{A},\mathcal{D}) = \boldsymbol{R}(\mathcal{A},Conn) = \{\Re\}$.

Assume that g is an arbitrary function and $\mathcal{F}$ is a family of functions from X into Y. We say that $A \subset X$ is *$(g,\mathcal{F})$-negligible* if every function $f : X \longrightarrow Y$ which coincides with g on $X \setminus A$ belongs to $\mathcal{F}$ (see [4] and [38]).

Theorem 8.3 *Let M be a subset of I. There exists an almost continuous function g such that M is a $(g,\mathcal{A}(I,\Re))$-negligible iff $I \setminus M$ is c-dense in I [38].*

Theorem 8.4 *Assume that g is an almost continuous real function defined on I. Then the following statements are equivalent:*

(i) $g \in \mathcal{D}^*(I,\Re)$,

(ii) every nowhere dense subset of I is $(g,\mathcal{A}(I,\Re))$-negligible,

(iii) there exists a dense subset of I which is $(g,\mathcal{A}(I,\Re))$-negligible [38].

Example 8.1 *There exists an almost continuous function $g : I \longrightarrow \Re$ such that all subsets of I which are small in the sense of cardinality (i.e. with the cardinality less than 2^ω) or of measure (i.e. of measure zero) or of category (i.e. of the first category) are $(g,\mathcal{A}(I,\Re))$-negligible.*

Indeed, as in the proof of Lemma 6.1 one can construct a function $g \in \mathcal{A}(I,\Re)$ such that $card(P \cap dom(K \cap g)) = 2^\omega$ for each minimal blocking set K and every non-empty perfect set $P \subset dom(K)$. Then g is OK.

Theorem 8.5 *Suppose that* $f, g \in \mathcal{D}^*(I, I)$ *and there exists a finite subset* A *of* I *such that* $f^{-1}(y) = g^{-1}(y)$ *for all* $y \in I \setminus A$. *Then* f *and* g *are both almost continuous or both not almost continuous [38].*

Recall that a class $\mathcal{F}$ of real functions is said to be characterizable by associated sets if there exists a family of sets $\mathcal{P}$ so that $f \in \mathcal{F}$ iff for all $y \in \Re$ the sets $[f < y]$ and $[f > y]$ belong to $\mathcal{P}$ [8].

Corollary 8.4 *The class* $\mathcal{A}(I, \Re)$ *is not characterizable by associated sets [38].*

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