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A Fuge on Bernstein Sets

Abstract. A subset B of a Polish space X is called *Bernstein* set if both B and $X \setminus B$ meet every uncountable, closed subset of X. Such sets neither are Lebesgue measurable nor have the property of Baire, and as such, are of interest to analysts, topologists and descriptive set theorists alike. Taking advantage of the algebraic structure of $\mathbb{R}^k$, we exhibit a family of $2^{\mathfrak{c}}$ homogeneous Bernstein sets in $\mathbb{R}$. We also show that if X and Y are uncountable Polish spaces and $f : X \to Y$ is continuous, then Bernstein sets are preserved under inverse images if and only if f is at most ω-to-1.

0. Prelude. We assume the axiom of choice holds. A subset B of a Polish space X is called *Bernstein* set if both B and $X \setminus B$ meet every uncountable, closed subset of X. In [1], Kuratowski proved that there exist uncountable, rigid (*i.e.*, admitting the identity as the only self-homeomorphism) subsets of $\mathbb{R}$. I have been informed, though I do not know by whom, that his technique has been used to construct a nonhomogeneous Bernstein set. Since there are $2^{\mathfrak{c}}$ Bernstein sets in $\mathbb{R}$, there must be $2^{\mathfrak{c}}$ nonhomogeous Bernstein sets in $\mathbb{R}$. For if B_1 is a Berstein set and B_2 is a nonhomogeneous Bernstein set, then $[B_1 \cap (-\infty, 0)] \cup [B_2 \cap (0, \infty)]$ is a nonhomogeneous Bernstein set. We will show, on the other hand, that there are $2^{\mathfrak{c}}$ homogeneous Bernstein sets in $\mathbb{R}$ by constructing a family of $2^{\mathfrak{c}}$ additive subgroups of $\mathbb{R}$, each of which is also a Bernstein set.

Given a Polish space X, if card $(X) \leq \omega$, then every subset of X is a Bernstein set. To see that Bernstein sets exist in uncountable Polish spaces, well order the set of uncountable closed subsets of such a space X as $\{C_\alpha | \alpha < \mathfrak{c}\}$, and then proceed inductively to choose a pair of points from each C_α, one to be in B and one not to be in B. (Ah, Shakespeare; but see [3] for a discussion of Bernstein sets.) We will begin by generalizing the notion of a Bernstein set, and then will consider these generalized Bernstein sets in the presence of certain algebraic structures.

Let X be a set and let $\mathcal{F} \subset \mathcal{P}(X)$. A set $B \subset X$ such that both B and $X \setminus B$ meet every $F \in \mathcal{F}$ will be called an *$\mathcal{F}$-Bernstein* set; equivalently, neither B nor $X \setminus B$ contains an element of $\mathcal{F}$. For example, if X is a space and $\mathcal{F}$ is the collection of all nonvoid open subsets of X, then the $\mathcal{F}$-Bernstein sets are precisely those dense subsets of X having dense complements. If for some $F \in \mathcal{F}$, $\operatorname{card}(F) < 2$ then there can be no $\mathcal{F}$-Bernstein set. On the other hand, if $\operatorname{card}(\cap\mathcal{F}) \geq 2$, or if $\mathcal{F}$ is a pairwise disjoint family of sets, each with cardinality ≥ 2, then there exists an $\mathcal{F}$-Bernstein set. Now suppose $g : X \to X$ is a bijection such that for all $F \in \mathcal{F}$, $g^{-1}(F) \in \mathcal{F}$, and suppose B is an $\mathcal{F}$-Bernstein set. Then if $g(B)$ or $g(X \setminus B)$ contains some $F \in \mathcal{F}$ so does B or $X \setminus B$. Hence $g(B)$ is an $\mathcal{F}$-Bernstein set. Similarly, if for all $F \in \mathcal{F}$, $g(F) \in \mathcal{F}$ then $g^{-1}(B)$ is an $\mathcal{F}$-Bernstein set. Of course, in the case where X is a Polish space and $\mathcal{F}$ is the set of all uncountable closed subsets of X, B is simply a Bernstein set. We begin with a generalization of the result that every uncountable Polish space contains a Bernstein set. *Throughout this note, λ denotes a regular cardinal.*

0.0. Theorem. (Bernstein, 1908) Let X be a set and let $\mathcal{F} \subset \mathcal{P}(X)$ such that $\operatorname{card}(\mathcal{F}) = \lambda$. If for all $F \in \mathcal{F}$, $\operatorname{card}(F) \geq \lambda$ then there are $\geq 2^\lambda$ $\mathcal{F}$-Bernstein sets in X.

Proof: We repeat the proof of Bernstein's theorem except that we choose three points at each stage rather than two. Well order $\mathcal{F}$ as $\{F_\alpha | \alpha < \lambda\}$, and suppose $x_{\alpha,1}, x_{\alpha,2}$ and $x_{\alpha,3}$ have been chosen for all $\alpha < \beta < \lambda$. Choose three distinct points $x_{\beta,1}, x_{\beta,2}$ and $x_{\beta,3}$ in $F_\beta \setminus (\bigcup_{\alpha<\beta}\{x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}\})$. Set $B = \bigcup_{\alpha<\lambda}\{x_{\alpha,1}, x_{\alpha,2}\}$. Then B meets every $F \in \mathcal{F}$, and as $\bigcup\{x_{\alpha,3}\} \subset X \setminus B$, so does $X \setminus B$. Now let $A = \bigcup_{\alpha<\lambda}\{x_{\alpha,1}\}$ and note that for every $A' \subset A, B \setminus A'$ is an $\mathcal{F}$-Bernstein set. Since there are 2^λ subsets of A, there are $\geq 2^\lambda$ $\mathcal{F}$-Bernstein sets in X.

0.1. Corollary. Every uncountable Polish space contains $2^{\mathfrak{c}}$ Bernstein sets.

1. Fugue. Now let G be a group. If $S \subset G$ then $\langle S\rangle$ denotes the smallest subgroup of G containing S. For $A, B \subset G$, $AB = \{ab | a \in A, b \in B\}$. We write $\langle x\rangle$ for $\langle\{x\}\rangle$, $\langle S, x\rangle$ for $\langle S \cup \{x\}\rangle$, $\langle A, B\rangle$ for $\langle A \cup B\rangle$, αB for $\{a\}B$, etc.

We introduce two notions here. Let G be a group and let κ be the least cardinal such that for all $g \in G$ and $n \in \mathbb{N}$, $x^n = g$ has at most κ solutions. Then $\kappa - 1$ is the *torsion degree* of G. If κ is the least cardinal such that for all $g, a_2, \ldots, a_m \in G$, with $g \notin \langle a_2, \ldots, a_m\rangle$, and for all $m, n_1, \ldots, n_m \in \mathbb{N}$, the equation $x^{n_1}a_2x^{n_2}\cdots a_mx^{n_m} = g$ has most κ solutions, then $\kappa - 1$ is the *polytorsion degree* of G. We will denote the torsion degree of G by $\tilde{\tau}(G)$ and the polytorsion degree of G by $\tilde{\rho}(G)$. Note that if G is abelian, then $\tilde{\tau}(G) = \tilde{\rho}(G)$, and $\tilde{\tau}(G) = 0$ if and only if G is torsion free. Whether G is commutative or not, $\tilde{\tau}(G) \leq \tilde{\rho}(G)$. One of the referees has pointed out that "the semi-direct product of the free abelian group F on uncountably many generators x_α and the free abelian group on generators a, b, with actions $x_\alpha^{-1}ax_\alpha = b$ and $x_\alpha^{-1}bx_\alpha = a$ for each α ... has uncountable polytorsion degree, but countable torsion degree." Hence, the torsion degree and the polytorsion degree are not, in general, the same.

1.1. Lemma. Let G be a group of cardinality λ with $\rho(G) < \lambda$. Assume $\lambda > \omega$. Let A be a subgroup of G such that $\text{card}(A) < \lambda$ and suppose $B \subset$

$G \setminus A$ such that $\operatorname{card}(B) < \lambda$. Let $X \subset G \setminus (A \cup B)$ of cardinality λ. Then $\{\langle A, x\rangle | x \in X$ and $\langle A, x\rangle \cap B = \varnothing\}$ has cardinality λ.

Proof: Let S be the set of all finite strings of the form $x^{n_1} a_2 x^{n_2} \cdots a_m x^{n_m}$, and note that $\operatorname{card}(S) = \max\{\omega, \operatorname{card}(A)\} < \lambda$. For each $b \in B$ let $X(b) = \{x \in G | s = b; s \in S\}$, where $s = b$ means s thought of as a product. Then $\operatorname{card}(X(b)) \leq \max\{\tilde{\rho}(G), \omega, \operatorname{card}(A)\}$, and it follows that $\operatorname{card}(\cup_{b \in B} X(b)) < \lambda$. Now, $x \in X(b)$ if and only if $b \in \langle A, x\rangle$. Thus, there are $< \lambda$ $x \in X$ for which $\langle A, x\rangle \cap B \neq \varnothing$. Therefore, $\{\langle A, x\rangle | x \in X$ and $\langle A, x\rangle \cap B = \varnothing\}$ has cardinality λ.

1.2. Lemma. Suppose $\{G_\alpha | \alpha < \lambda\}$ is a collection of subgroups of G such that $\alpha < \beta \Longrightarrow G_\alpha \subset G_\beta$, then $\{\cup_{\alpha < \beta} G_\alpha\}$ is a collection of subgroups of G for all $\beta \leq \lambda$.

A subgroup of any group G which is also an $\mathcal{F}$-Bernstein set will be call an *$\mathcal{F}$-Bernstein* subgroup.

1.3. Theorem. Let G be a group of cardinality $\lambda > \omega$, and let $\mathcal{F} \subset \mathcal{P}(G)$ such that for each $F \in \mathcal{F}$, $\operatorname{card}(F) = \lambda$, and $\tilde{\rho}(G) < \lambda$. Then there is an $\mathcal{F}$-Bernstein subgroup of G.

Since the proof follows, *mutatis mutandis*, that of Theorem 4′ below, we omit it here.

1.4. Corollary. Every complete separable metric topological group G with $\tilde{\rho}(G) < \operatorname{card}(G)$ contains a Bernstein subgroup.

Let us say that a group G is *(left) $\mathcal{F}$-preservative* if for every $F \in \mathcal{F}$ and every $g \in G$, $gB \in \mathcal{F}$. We will say only "preservative" if $\mathcal{F}$ is the set of uncountable closed subsets. Note that by our earlier remarks, if G is $\mathcal{F}$-

preservative and B is an $\mathcal{F}$-Bernstein set, then gB is an $\mathcal{F}$-Bernstein set for all $g \in G$. Obviously $\mathbb{R}$ is preservative.

We now turn our attention to topological rings, and so, from now on, we will use juxtaposition to indicate multiplication in the ring. We assume that each ring in question contains a dense, isomorphic copy of $\mathbb{Q}^k$ for some $k < \omega$. Thus $\mathbb{Q}\langle S\rangle$ denotes the smallest subgroup of $(R, +)$ which contains S and is closed under multiplication by $q \in \mathbb{Q}$. It is easily verified that if $S = \mathbb{Q}\langle S\rangle$ then $\mathbb{Q}\langle S, r\rangle = S + \mathbb{Q}r$ for all $r \in R$. When we write $\tilde{\tau}(R)$, we mean $\tilde{\tau}((R, +))$.

For the rest of this section, we assume that $\text{card}(R) = \lambda > \omega$,and that $\mathcal{F} \subset \mathcal{P}(R)$ such that $\text{card}(\mathcal{F}) = \lambda$ and for each $F \in \mathcal{F}, \text{card}(F) = \lambda$.

1.1′ Lemma. Assume $\tilde{\tau}(R) < \lambda$. Let $A = \langle A\rangle \subset R$ with $\text{card}(A) < \lambda$, and suppose $B \subset R \setminus A$ with $\text{card}(B) < \lambda$. Let $X \subset R \setminus (A \cup B)$ with $\text{card}(X) = \lambda$. Then $\{\langle A, x\rangle | x \in X \text{ and } \langle A, x\rangle \cap B = \varnothing\}$ has cardinality λ.

1.2′ Lemma. If $\{S_\alpha | \alpha < \lambda\} \subset \mathcal{P}(R)$ such that $\alpha < \beta \Longrightarrow S_\alpha \subset S_\beta$ and for all $\alpha < \lambda$, $S_\alpha = \langle S_\alpha\rangle$, then $\cup_{\alpha<\beta} S_\alpha = \langle \cup_{\alpha<\beta} S_\alpha\rangle$. If in addition, each $S_\alpha = \mathbb{Q}\mathbb{S}_\alpha$, then $\cup S_\alpha = \mathbb{Q}(\cup \mathbb{S}_\alpha)$.

1.3′. Theorem. Let R be a ring of cardinality λ and let $\tilde{\tau}(R) < \lambda$. Let $\mathcal{F} \subset \mathcal{P}(R)$ such that $\text{card}(\mathcal{F}) = \lambda$ and for each $F \in \mathcal{F}$, $\text{card}(F) = \lambda$. Then there are $\geq 2^\lambda$ $\mathcal{F}$-Bernstein subgroups B of R such that $B = \mathbb{Q}B$. Moreover, there is a family $\{B_\alpha\}$ of λ distinct $\mathcal{F}$-Bernstein subgroups such that $\alpha < \beta \Longrightarrow B_\alpha \subset B_\beta$.

Proof: Let $\{F_\alpha | \alpha < \lambda, \lambda \text{ even}\}$ be a well-ordering of $\mathcal{F}$. Let $x_0 \in F_0$ and set $Q_0 = \mathbb{Q}x_0$. Then let $x_1 \in F_0 \setminus Q_0$ and set $Q_1 = Q_0 + \mathbb{Q}x_1$. Let $\bar{x}_0 \in F_0 \setminus Q_1$ and set $P_0 = P_1 = \{\bar{x}_0\}$. Suppose Q_α and P_α have been chosen for all $\alpha < \beta < \lambda$, β even. Let $x_\beta \in F_\beta \setminus ((\cup_{\alpha<\beta} Q_\alpha) \cup (\cup_{\alpha<\beta} P_\alpha))$ such that $[\cup_{\alpha<\beta}(Q_\alpha + \mathbb{Q}x_\beta)] \cap [\cup_{\alpha<\beta} P_\alpha] = \varnothing$ and set $Q_\beta = \cup_{\alpha<\beta} Q_\alpha + \mathbb{Q}x_\beta$. Such an x_β exists by Lemma 1.1$'$. Next let $x_{\beta+1} \in F_\beta \setminus (Q_\beta \cup (\cup_{\alpha<\beta} P_\alpha))$ such that $(Q_\beta + \mathbb{Q}x_{\beta+1}) \cap (\cup_{\alpha<\beta} P_\alpha) = \varnothing$ and set $Q_{\beta+1} = Q_\beta + \mathbb{Q}x_{\beta+1}$. Then let $\bar{x}_\beta \in F_\beta \setminus Q_{\beta+1}$ and set $P_\beta = P_{\beta+1} = (\cup_{\alpha<\beta} P_\alpha) \cup \{\bar{x}_\beta\}$. Take $B = \cup_{\alpha<\lambda} Q_\alpha$. Clearly B is an $\mathcal{F}$-Bernstein set. By Lemma 1.2$'$, $B = \mathbb{Q}B$.

Let A be the set of odd ordinals $< \lambda$, and let $A' \subset A$. For all $\beta < \lambda$, let $x'_\beta = x_\beta$ for $\beta \in \lambda \setminus A'$ and let $x'_\beta = 0$ for $\beta \in A'$. Modify the proceeding induction scheme as follows. Set $Q'_0 = Q_0$ and define $Q'_\beta = (\cup_{\alpha<\beta} Q'_\alpha) + \mathbb{Q}x'_\beta$, while keeping everything else the same. Then $B' = \cup_{\alpha<\lambda} Q'_\alpha$ is an $\mathcal{F}$-Bernstein subgroup of R since for every even ordinal β, $x'_\beta \in B' \cap F_\beta$ and $\bar{x}_\beta \in F_\beta \setminus B'$. Now let $\beta \in A'$. If $x_\beta \in B'$ then there is a least ordinal γ such that $x_\beta \in Q'_\gamma$. We must have $x_\beta = a + qx_\gamma$ for some $a \in \cup_{\alpha<\gamma} Q'_\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$. However, if $\gamma < \beta$ then $x_\beta \in \cup_{\alpha<\gamma} Q_\alpha$, contradicting the original choice of x_β. On the other hand, if $\beta < \gamma$ then $x_\beta - a = qx_\gamma \Longrightarrow \frac{1}{q}(x_\beta - a) = x_\gamma \in \cup_{\alpha<\gamma} Q_\alpha$ which is also a contradiction. Therefore, $B' \cap \{x_\beta | \beta \in A'\} = \varnothing$. Since there are 2^λ subsets of A, there are $\geq 2^\lambda$ $\mathcal{F}$-Bernstein subgroups B of R such that $B = \mathbb{Q}B$.

Now let B_0 be the $\mathcal{F}$-Bernstein subgroup of R formed by taking $A' = A$ above. Suppose B_α has been chosen for all $\alpha < \beta < \lambda, \beta$ even. Set $B_{\beta+1} = \langle (\cup_{\alpha<\beta} B_\alpha), x_{\beta+1} \rangle$. Then $\{B_0, B_1, B_3, \cdots, B_{\omega+1}, \cdots\}$ is an increasing family of λ $\mathcal{F}$-Bernstein subgroups of R. We reason as in the proceeding paragraph to show that $\alpha \neq \beta \Longrightarrow B_\alpha \neq B_\beta$. If $\alpha < \beta$, $\beta \in A$, and $x_\beta \in \cup_{\alpha<\beta} B_\alpha$, then $x_\beta = p_1 x_{\alpha_1} + \cdots + p_m x_{\alpha_m} + q_1 x_{\beta_1} + \cdots + q_n x_{\beta_n}$, where for all i, α_i is even, β_i is odd and $p_i, q_i \in \mathbb{Q}$, and $\alpha_1 < \cdots < \alpha_m$, $\beta_1 < \cdots < \beta_n$. Clearly $\beta_n < \beta$. If $\alpha_m < \beta$ then the equation can not hold (see the preceding paragraph). If $\alpha_m > \beta$, then $p_m x_{\alpha_m} = x_\beta - p_1 x_{\alpha_1} - \cdots - p_{m-1} x_{\alpha_{m-1}} - q_1 x_{\beta_1} - \cdots - q_n x_{\beta_n} \in \cup_{\alpha<\alpha_m} Q_\alpha$ which is also impossible.

1.4$'$. Corollary. Every perfect complete separable metric topological ring containing a dense isomorphic copy of $\mathbb{Q}^k$ where $k < \omega$ (such as $\mathbb{R}$ or $\mathbb{C}$) contains $2^{\mathfrak{c}}$ Bernstein subgroups B such that $B = \langle B \rangle$. Moreover, there are

$2^{\mathfrak{c}}$ families $\{B_\alpha | \alpha < 2^\omega\}$ of Bernstein sets with $B_\alpha = \langle B_\alpha \rangle$ such that if $\alpha < \beta$ then $B_\alpha \subset B_\beta$.

2. Coda. We include one more result.

2. Theorem. Let X and Y be uncountable Polish spaces and let $f : X \to Y$ be continuous. Then for every Bernstein set $B \subset Y$, $f^{-1}(B)$ is a Bernstein set in X if and only if for every $y \in Y$, $\text{card}(f^{-1}(y)) \leq \omega$.

Proof: if) Assume the hypotheses, let $B \subset Y$ and suppose there is an uncountable closed subset $C \subset f^{-1}(B)$. Then $f(C) \subset B$ and $f(C)$ is analytic. Moreover, $f(C)$ is uncountable because f is at most ω-to-1. Since every uncountable analytic set contains a Cantor set, B is not a Bernstein set.

only if) Suppose for some $y \in Y$, $\text{card}(F^{-1}(y)) > \omega$. Let B be a Bernstein set in Y such that $y \in B$. Then $f^{-1}(B)$ contains the uncountable closed set $f^{-1}(y)$ and so is not a Bernstein set.

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