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# DERIVATIVES AND THE CARATHÉODORY SUPERPOSITION

Let $\mathbf{R}$ be the set of reals. The density topology $T_d$ ([1], [8], [10]) on $\mathbf{R}$ consists of all measurable subsets $A$ of $\mathbf{R}$ such that, for every $x \in A$, $x$ is a density point of $A$. Let $I \subset \mathbf{R}$ be an interval. A function $f : I \to \mathbf{R}$ is density continuous ([5], [6], [7]) if it is continuous as a map from $(I, T_d)$ into $(R, T_d)$.

A family $\mathcal{F}$ of maps of the topological space $(\mathbf{R}, T_d)$ into $\mathbf{R}$ (with the natural topology) is said to be $T_d$-equicontinuous at a point $x \in \mathbf{R}$ ([9], p. 188), if, given $\varepsilon > 0$, there is a neighborhood $V \in T_d$ of $x$ such that $|f(u) - f(x)| < \varepsilon$ for each $u \in V$ and $f \in \mathcal{F}$. We say that $\mathcal{F}$ is $T_d$-equicontinuous on $\mathbf{R}$ if it is $T_d$-equicontinuous at each point.

In the paper [2] I proved the following theorem:

<u>**Theorem 0.**</u> Suppose that $D \subset \mathbf{R}^2$ is a nonempty open set and $f : D \to \mathbf{R}$ is a locally bounded function such that all sections $f^y(t) = f(t, y)$ ($t, y \in \mathbf{R}$ and $(t, y) \in D$) are derivatives and all sections $f_x(t) = f(x, t)$ ($x, t \in R$ and $(x, t) \in D$) are equicontinuous. Then for every continuous function $g : I \to \mathbf{R}$ such that $(x, g(x)) \in D$ for $x \in I$ and $I$ is an interval, the function $h(x) = f(x, g(x))$ is a derivative.

In this paper we approach the derivative structure of the function $h$ in terms of density continuity.

<u>**Theorem 1.**</u> Suppose that $D \subset \mathbf{R}^2$ is a nonempty open set and $f : D \to \mathbf{R}$ is a locally bounded function such that all sections $f^y$ are derivatives and all sections $f_x$ are $T_d$-equicontinuous. Then for every continuous and density continuous function $g : I \to \mathbf{R}$ such that $I$ is an interval and $(x, g(x)) \in D$ for $x \in I$, the superposition $h(x) = f(x, g(x))$ is a derivative.

<u>**Proof.**</u> First, we remark that the function $h$ is measurable in the Lebesgue sense ([4]). We shall prove that $h$ is a derivative at each point $x \in I$, i.e.

$$\lim_{t\to x}\int_x^t h(u)du/(t-x)=h(x).$$

Fix $x\in I$ and $\varepsilon>0$. Since $f$ is locally bounded and $g$ is continuous at $x$, there are numbers $s,s_1>0$ such that $s_1<s$,

$$(1)\qquad M = \sup\{|f(u,v)|:|u-x|\le s,|v-g(x)|\le s\}<\infty,$$

$$\text{and } |g(t)-g(x)|<s \ \text{ for } \ t\in I\cap(x-s_1,x+s_1).$$

Without loss of generality we may assume that $M>0$ and $[x,x+s]\subset I$. Since the section $f^{g(x)}$ is a derivative at $x$, we have

$$\lim_{t\to x}\int_x^t f(u,g(x))du/(t-x)=f(x,g(x)).$$

Consequently, there is $t_0>x$ such that, $t_0<x+s_1$ and

$$(2)\qquad \left|\int_x^t f(u,g(x))du/(t-x)-f(x,g(x))\right|<\varepsilon/3 \ \text{ for } \ x<t\le t_0.$$

Since all sections $f_u$ are $T_d$-equicontinuous, there exists a $T_d$-neighborhood $V\in T_d$ of $g(x)$ such that, $V\subset(g(x)-s,g(x)+s)$, and

$$(3)\qquad |f(u,y)-f(u,g(x))|<\varepsilon/3 \ \text{ for each } \ u\in I,\ y\in V.$$

Observe that $g^{-1}(V)\in T_d$ and $x$ is a density point of $g^{-1}(V)$. There is $t_1>x$ such that, $t_1<t_0$, and

$$(4)\qquad m((x,t)\cap g^{-1}(V))/(t-x)>1-\varepsilon/6M$$

for $x<t\le t_1$ ($m$ denotes Lebesque measure). If $x<t<t_1$, then

$$
(5)\qquad \begin{cases}
|\int_x^t h(u)du/(t-x) - h(x)| = \\
|\int_x^t f(u,g(u))du/(t-x) - f(x,g(x))| = \\
|\int_x^t (f(u,g(u)) - f(x,g(x)))du/(t-x)| = \\
|\int_x^t (f(u,g(u)) - f(u,g(x)))du/(t-x) + \\
\int_x^t (f(u,g(x)) - f(x,g(x)))du/(t-x)| \leq \\
\int_x^t |f(u,g(u)) - f(u,g(x))|du/(t-x) + \\
|\int_x^t (f(u,g(x)) - f(x,g(x)))du/(t-x)|.
\end{cases}
$$

Using (2) we have for $x < t < t_1$ the following:

$$
(6)\qquad \begin{aligned}
&|\int_x^t f(u,g(x)) - f(x,g(x)))du/(t-x)| = \\
&|\int_x^t f(u,g(x))du/(t-x) - f(x,g(x))| < \varepsilon/3.
\end{aligned}
$$

On the other hand, from (1), (3) and (4) we have for $x < t < t_1$,

$$
(7)\qquad \begin{cases}
\int_x^t |f(u,g(u)) - f(u,g(x))|du = \\
\int_{(x,t)\cap g^{-1}(V)} |f(u,g(u)) - f(u,g(x))|du + \\
\int_{(x,t)\setminus g^{-1}(V)} |f(u,g(u)) - f(u,g(x))|du \leq \\
\varepsilon(t-x)/3 + 2M\ \varepsilon(t-x)/6M = 2\varepsilon(t-x)/3.
\end{cases}
$$

Thus, from (5), (6) and (7) we obtain

$$
\left|\int_x^t h(u)du/(t-x) - h(x)\right| < \varepsilon
$$

and consequently,

$$
\lim_{t\to x^+} \int_x^t h(u)du/(t-x) = h(x).
$$

Analogously we may show that

$$
\lim_{t\to x^-} \int_x^t h(u)du/(t-x) = h(x).
$$

This completes the proof.

**Remark.** In my paper [3] I proved that there is a function $f : \mathbf{R}^2 \to [0,1]$ such that all its sections $f_x, f^y$ are continuous, $f(x,x) = 1$ for $x > 0$ and $f(0,0) = 0$. Note that the function $g(x) = x$ is continuous and density continuous, but the superposition $h(x) = f(x, g(x))$ is not a derivative.

**Theorem 2.** Let $L > 0$ be a constant. There are a function $g : \mathbf{R} \to \mathbf{R}$ satisfying the Lipschitz condition with constant $L$ and a function $f : \mathbf{R}^2 \to [0,1]$ having all constant sections $f^y$ and all $T_d$-equicontinuous sections $f_x$ such that the superposition $h(x) = f(x, g(x))$ is not a derivative.

**Proof.** Let $(b_n)_{n=0}$ be a decreasing sequence such that $\lim_{n\to\infty} b_n = 0$ and $b_n > 0$, $n = 1, 2, \ldots$ and

$$(b_{n-1} - b_n)2^{2n-2} < L \text{ for } n = 1, 2, \ldots \tag{8}$$

(for example $b_n = L/(8n)!$). Put

$$g(x) = \begin{cases} b_0 & \text{for } x \geq 1 \\ b_n & \text{for } x \in [2^{-2n}, 2^{-2n+1}],\ n = 1, 2, \ldots \\ 0 & \text{for } x \leq 0 \\ \text{linear on the intervals } [2^{-2n+1}, 2^{-2n+2}],\ n = 1, 2, \ldots. \end{cases}$$

From (8) it follows that $g$ satisfies the Lipschitz condition with the constant $L$. There are intervals $I_n = [b_n - a_n, b_n + a_n]$ $(n = 0, 1, \ldots)$ such that $I_n \cap I_k = \phi$ for $k \neq n$ and 0 is a density point of the set

$$E = \mathbf{R} \setminus \bigcup_{n=0}^{\infty} I_n.$$

Let $z : \mathbf{R} \to [0,1]$ be continuous at each point $x \neq 0$ and such that $z(b_n) = 1$ for $n = 0, 1, \ldots$ and $z(x) = 0$ for $x \in E$. Define

$$f(x,y) = z(y) \text{ for } (x,y) \in \mathbf{R}^2.$$

All sections $f^y$ are constant and all sections $f_x$ are $T_d$-equicontinuous. Moreover, the function $h(x) = f(x, g(x))$ is such that $h(0) = 0$ and $h(x) = 1$ for $x \in [2^{-2n}, 2^{-2n+1}]$, $n = 1, 2, \ldots$. Since 0 is not a density point of the set

$$\mathbf{R} \setminus \bigcup_{n=1}^{\infty} [2^{-2n}, 2^{-2n+1}]$$

and $h(x) \geq 0$ for all $x \in \mathbf{R}$, $h$ is not a derivative at 0.

<u>**Theorem 3.**</u> Let $L > 0$ be a constant. There are a function $g : \mathbf{R} \to \mathbf{R}$ satisfying the Lipschitz condition with constant $L$ and a function $f_1 : \mathbf{R}^2 \to [0,1]$ having continuous sections $(f_1)_x, (f_1)^y$ and $T_d$-equicontinuous sections $(f_1)_x$, such that the superposition $h(x) = f(x, g(x))$ is not a derivative.

<u>**Proof.**</u> Let $f, g$ be the same as in the proof of Theorem 2. The sets

$$A = \{(x,y) : x > 0,\ y \leq g(x)/4\},$$

$$B = \{(x,y) : x > 0,\ y \geq g(x)/2\}$$

are closed in the space $C = \{(x,y) \in \mathbf{R}^2 : x > 0\}$ with the natural topology. There exists a continuous function $f_2 : C \to [0,1]$ such that $f_2(x,y) = 0$ for $(x,y) \in A$ and $f_2(x,y) = 1$ for $(x,y) \in B$. Define

$$f_1(x,y) = \begin{cases} f_2(x,y)f(x,y) & \text{for } (x,y) \in C \\ 0 & \text{otherwise.} \end{cases}$$

All sections $(f_1)_x, (f_1)^y$ are continuous, the sections $(f_1)_x$ are $T_d$-equicontinuous and the superposition $h(x) = f_1(x, g(x))$ is not a derivative.

<u>**Example.**</u> Let $I_n = [a_n, b_n]$, $n = 1, 2, \ldots$, be a sequence of intervals such that $a_1 < b_1 < a_2 < b_2 < \cdots \to 0$ and 0 is a density point of the set $\mathbf{R} \setminus \bigcup_{n=1}^{\infty} I_n$. For each $n = 1, 2, \ldots$ there exists a closed interval $J_n \subset (a_n, b_n)$. There is a function $f : \mathbf{R}^2 \to [0,1]$ continuous at each point $(x,y) \neq (0,0)$ such that

$$f(x,y) = \begin{cases} 0 & \text{if } x \in \mathbf{R} \setminus \bigcup_{n=1}^{\infty} I_n, \\ 0 & \text{if } x \in I_n \text{ and } y \leq 1/2n,\ n = 1, 2, \ldots, \\ 1 & \text{if } x \in J_n \text{ and } y \geq 1/n,\ n = 1, 2, \ldots. \end{cases}$$

Then the sections $f_x (x \in \mathbf{R})$ are not $T_d$-equicontinuous, but for every continuous function $g : I \to \mathbf{R}$, where $I$ is an interval, the function $h(x) = f(x, g(x))$ is a derivative.

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