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STOCHASTIC AND OTHER FUNCTIONAL INTEGRALS

Given a fixed real number a, the space T of all real valued functions f on $[a,\infty)$ with $f(a)=0$, and a fixed $F \in T$, we consider integrals

$$\int_a^\infty F(t)df(t) \quad (f \in T). \tag{1}$$

When $F \in L^2(a,\infty)$, T.W. Lee [8] defined (1) as an integral equivalent to that of Paley, Wiener and Zygmund [9], that exists for all $f \in T$ except a set of Wiener measure zero, so restricting f to be continuous. An integral like that of Itô [6] would need $F(t,f)$ for $F(t)$; this will be discussed elsewhere. The present paper fills a gap in [8], uses a more general measure than Wiener's, and gives properties of the integral.

The construction uses gauge integrals, the latest book discussing them being [4], with a brief history on *p.* 196. Briefly, let $-\infty \le u < v \le +\infty$, let $\delta(t) > 0$ for all finite $t \in [u,v]$ (δ is called a *gauge*), and let real numbers A, B be given, respectively when $u=-\infty$, $v=+\infty$. Then the interval-point pair $([r,s),t)$ is (A,B,δ)-fine if either $t=r$ or s with $s-r<\delta(t)$, or $s<A$ when $r=-\infty=u$, or $r>B$ when $s=+\infty=v$. A finite collection of interval-point pairs $([r,s),t)$ is (A,B,δ)-fine if each pair is (A,B,δ)-fine, and the collection is a *division* $\mathcal{D}$ of $[u,v)$ if the intervals are disjoint with union $[u,v)$. If $u>-\infty$ we omit A, if $v<+\infty$ we omit B, and if u,v are both finite we write "δ-fine". The generalized intervals $I \subseteq T$ of real valued functions are Cartesian products $\bigotimes I(t)$ of one-dimensional finite or infinite intervals $I(t)$ for all $t>a$, where for some finite subset $C \subset (a,\infty)$,

$$I(t) \subseteq (-\infty,\infty)\ (t \in C),\ I(t) = (-\infty,\infty)\ (t \notin C,\ t>a).$$

Note that $t=a$ is omitted since $f(a)=0$ and we would need $I(a)$ to be just a. For a given I let $C(I)$ be the smallest C, so that for $t \in C(I)$, $I(t) \ne (-\infty,\infty)$. Let x_b be the variable, $[u_b,v_b)$ an interval, $\delta_b(f^C;x_b)>0$ a gauge, with suitable A_b, B_b, and let $([u_b,v_b),x_b)$ be (A_b,B_b,δ_b)-fine, all for $t=b\in C$. Here, f^C is the finite set of values $f(b)$ for $b \in C$. As C is a finite set we can replace δ_b by $\delta_C(f^C;x_b) \equiv \min_{b\in C} \delta_b(f^C;x_b) > 0$, *with* $A^C = \min_{b\in C} A_b, B^C = \min_{b\in C} B_b$. For each

$f \in T$ let $C_1(f) \subset (a, \infty)$ be a finite set. Then we restrict C to be a finite set satisfying $C_1(f) \subseteq C \subset (a, \infty)$. Given the $C_1(f), C, \delta_C, A^C, B^C$, there are divisions $\mathcal{D}$ of I, so that we can define the *generalized Riemann integral* $H(I)$ of an interval-point function $h(J, f)$ to be a value such that for each $\varepsilon > 0$ and some $C_1(f), C, \delta_C, A^C, B^C$, and all (A^C, B^C, δ_C)-fine divisions $\mathcal{D}$ of I,

$$|(\mathcal{D})\Sigma h(J, f) - H(I)| < \varepsilon.$$

We then have the usual Fubini-type theorems, see [2], [3], [5]. The most general form of interval-point function for which Fubini's theorem holds when the integral over T exists, for all partitions of (a, ∞) into a finite set and its complement, is one variationally equivalent to

$$h(I, f) \equiv g(C(I); f) \prod_{b \in C(I)} k^b(I(b), f(b)) \tag{2}$$

for some g, k^b. See [3]. Sometimes, for

$$C = (b_1, \ldots, b_n),\ a = b_0 < b_1 < \ldots < b_n, \tag{3}$$
$$g(C, f) \equiv Q(f) \prod_{j=1}^{n} q(f(b_j) - f(b_{j-1});\ b_j - b_{j-1}),$$

$$k^b([u, v), f(b)) \equiv v - u,\ Q(f) \equiv 1. \tag{4}$$

Wiener measure results from (2), (3), (4) with

$$q(x; c) = (2\pi K c)^{-\frac{1}{2}} \exp(-x^2/(2Kc))\ \ (c > 0)$$

and $K = \frac{1}{2}$, while one of the Feynman measures has $K = \frac{1}{2}i$. See [3], *p.* 218; [5], *Chapter* 8.

For each $b > a$ let $I(b) = [u_b, v_b)$, let $I = \bigotimes_{b > a} I(b)$, $C(I) \subseteq C \subset (a, \infty)$, C finite, with $H_C(I)$ the integral of h over the Cartesian product of $I(b)$ for $b \in C$. Then the Smoluchowski-Chapman-Kolmogorov relation for h is that $H_C(I)$ is independent of $C \supseteq C(I)$. Thus we omit C and write $H(I)$. If $b > a$, $b \notin C(I)$,

$$\int_{-\infty}^{\infty} g(C; f) dk^b(I(b), f(b)) = g(C(I); f)\ (C = C(I) \cup \operatorname{sing}(b)), \tag{5}$$

for fH-almost everywhere in T. In (3), (4), for almost all $f(b_n), f(b_{j+1}), f(b_j)$,

$$\int_{-\infty}^{\infty} q(f(b) - f(b_n); b - b_n) df(b) = 1\ \ (b > b_n),$$

$$\int_{-\infty}^{\infty} q(f(b) - f(b_j); b - b_j) q(f(b_{j+1}) - f(b); b_{j+1} - b) df(b)$$
$$= q(f(b_{j+1}) - f(b_j); b_{j+1} - b_j)(b_j < b < b_{j+1}).$$

For $x = f(b) - f(b_k)$, $c = b - b_k(k = n$ or $j)$, $c' = b_{j+1} - b$, and almost all $y = f(b_{j+1}) - f(b_j)$,

$$\int_{-\infty}^{\infty} q(x;c)dx = 1 \ (c > 0), \tag{6}$$

$$\int_{-\infty}^{\infty} q(x;c)q(y-x;c')dx = q(y;c+c') \ \ (c > 0,\ c' > 0).$$

The second is a convolution.

Next we have an extension of the gauge integral theory of [4].

Lemma 1. *If the gauge integral M of an $m(I,x)$ exists in $[a,w)$ for all finite $w > a$, then, given $\varepsilon > 0$, a gauge $\delta > 0$ exists such that for all δ-fine divisions $\mathcal{D}$ of $[a,\infty)$,*

$$(\mathcal{D})\Sigma'|M(I) - m(I,t)| < \varepsilon,$$

where in Σ' we omit the term for $([u,\infty),\infty) \in \mathcal{D}$.

Proof. See [4], *p.* 53, *Theorem* 5.3, for $[a+j-1, a+j)$, and $\varepsilon.2^{-j}$ for $\varepsilon(j = 1,2,\ldots)$. This does not need B nor the integral over $[a,\infty)$.

T.W. Lee [8] used P.Y. Lee [7], *Lemma* 4, *p.* 315, as follows.

If $F \in L^2(a,\infty)$ and $\varepsilon > 0$, there is a step function $G(x;\varepsilon)$ with

$$||F - G||_2 \equiv \sqrt{\int_a^{\infty} |F(x) - G(x;\varepsilon)|^2 dx} < \varepsilon.$$

This is not strong enough for T.W. Lee's purpose, we need $G(x;\varepsilon)$ to depend on divisions as well as ε. We define, for $\mathcal{D}$ a division of $[a;\infty)$, $F_{\mathcal{D}}(x) = F(t)$ $(u \le x < v < \infty$, $([u,v),t) \in \mathcal{D})$; otherwise $F_{\mathcal{D}}(x) = 0$.

Lemma 2. For $F \in L^2(a,\infty)$, hence F measurable, given $\varepsilon > 0$, a gauge $\delta > 0$ on $[a,\infty)$ and a constant $B > a$ exist so that for all (B,δ)-fine divisions $\mathcal{D}$ of $[a,\infty)$,

$$||F - F_{\mathcal{D}}||_2 < \varepsilon.$$

Proof. For real-valued F and finite $w > a$, by the Cauchy-Buniakowski-Schwarz inequality and the measurability of F, $|F| \in L^1(a,w)$, and so $F \in L^1(a,w)$. Writing

$$G(u,v) \equiv \int_u^v F\,dx \ (a \le u < v < \infty),$$

given $\varepsilon > 0$, by Lemma 1 $\delta_j(x) > 0$ exist on $[a,\infty)$ such that for all δ_j-fine divisions $\mathcal{D}$ of $[a,\infty)$,

$$(\mathcal{D})\Sigma'|F(t)(v-u) - G(u,v)| < \varepsilon.4^{-j} \ \ (j = 1,2,\ldots). \tag{7}$$

For some gauge $\delta_0 > 0$ on $[a, \infty)$ and $B > a$, and all (B, δ_0)-fine divisions $\mathcal{D}$ of $[a, \infty)$,

$$(8) \qquad |(\mathcal{D})\Sigma' F^2(t)(v-u) - \int_a^\infty F^2 dx| < \varepsilon.$$

For X_j the set of x with $|F(x)| \leq 2$ $(j = 1)$, $2^{j-1} < |F(x)| \leq 2^j$ $(j = 2, 3, \ldots)$, let $\delta(x) \equiv \min(\delta_j(x), \delta_0(x)) > 0 (x \in X_j, j = 1, 2, 3, \ldots)$ with $\mathcal{D}$ a (B, δ)-fine division of $[a, \infty)$. For w the largest finite v in $\mathcal{D}$, with (8), (7),

$$(9) \qquad \int_a^w F_{\mathcal{D}}^2 dx = (\mathcal{D})\Sigma' F^2(t)(v-u), |\int_a^\infty F^2 dx - \int_a^w F_{\mathcal{D}}^2 dx| < \varepsilon,$$

$$\int_a^w F_{\mathcal{D}}(F - F_{\mathcal{D}}) dx = (\mathcal{D})\Sigma' \int_u^v F(t)(F - F(t)) dx$$

$$= (\mathcal{D})\Sigma' F(t)\{G(u,v) - F(t)(v-u)\},$$

$$(10) \qquad |\int_a^w F_{\mathcal{D}}(F - F_{\mathcal{D}}) dx| < \sum_{j=1}^{\infty} 2^j \varepsilon 4^{-j} = \varepsilon.$$

For real valued F the result follows from (8), (9), (10) and

$$\int_a^\infty (F - F_{\mathcal{D}})^2 dx = \int_a^\infty F^2 dx - \int_a^w F_{\mathcal{D}}^2 dx - 2\int_a^w F_{\mathcal{D}}(F - F_{\mathcal{D}}) dx,$$

$$0 \leq \int_a^\infty (F - F_{\mathcal{D}})^2 dx < 3\varepsilon.$$

For complex valued F the real and imaginary parts lie in $L^2(a, \infty)$ and

$$(11) \qquad ||F - F_{\mathcal{D}}||_2^2 < 6\varepsilon.$$

Lemma 3. *For a finite subset $C \subset (a, \infty)$ let $T(C)$ be the Cartesian product of $I(b) = (-\infty, \infty)(b \in C)$. Let $F : T \to R$ be a functional F_1 that is constant relative to $f(b)$ for $b \notin C$, with H the integral of h, and the Smoluchowski-Chapman-Kolmogorov relation. If $F(f)H(I)$ is integrable over T, so is $F_1(f)H(I)$ over $T(C)$ and*

$$\int_T F dh = \int_{T(C)} f_1 dH.$$

This is Wiener's formula when using Lebesgue integrals and Wiener measure.

Proof. Use [3], *pp.* 223-4, *Theorem* 5(15).

To extend T.W. Lee [8], *p.* 66, *Proposition*, we use (4) and generalize g in (3) to

$$r(f(b_1)-f(b_0), b_1-b_0, f(b_2)-f(b_1), \\ b_2-b_1,\ldots, f(b_n)-f(b_{n-1}), b_n-b_{n-1}) \geq 0. \tag{12}$$

For simplicity we write $r_j(x;c)$ for (12) with $f(b_j)-f(b_{j-1})=x, b_j-b_{j-1}=c>0$, keeping $f(b_k)-f(b_{k-1})$ and b_k-b_{k-1} fixed $(k \neq j)$, where for some $K>0$,

$$\int_{-\infty}^{\infty} x r_j(x;c)dx = 0, \quad \int_{-\infty}^{\infty} x^2 r_j(x;c)dx \leq Kc. \tag{13}$$

THEOREM 1. *Let $H \geq 0$ be the integral in $T(C)$ of the h of* (2), (4), (5), (12), (13). *Writing $\Delta f \equiv f(v)-f(u)$, for $F \in L^2(a,\infty)$ and two divisions $\mathcal{D}, \mathcal{E}$* of $[a,\infty)$,

$$||(\mathcal{D})\Sigma' F\Delta f - (\mathcal{E})\Sigma' F\Delta f|| \equiv \sqrt{\int_T |(\mathcal{D})\Sigma' F\Delta f - (\mathcal{E})\Sigma' F\Delta f|^2 dH} \\ \leq \sqrt{K}||F_{\mathcal{D}} - F_{\mathcal{E}}||_2; \tag{14}$$

$$||(\mathcal{D})\Sigma' F\Delta f|| \leq \sqrt{K}||F_{\mathcal{D}}||_2. \tag{15}$$

A functional $k(F;f)$ is uniquely defined H-almost everywhere in T with the following property. Given $\varepsilon>0$, there are a gauge $\delta>0$ on $[a,\infty)$ and a number $B>a$ with

$$||(\mathcal{D})\Sigma' F\Delta f - k(F;f)|| < \varepsilon \tag{16}$$

for all (B,δ)-fine divisions $\mathcal{D}$ of $[a,\infty)$.

Corollary 1. *If, for the f in a set of non-zero H-variation,* (1) *exists as a generalized Riemann integral, its value is $k(F;f)$ H-almost everywhere in the set.*

Corollary 2. *If equality in* (15), *$k(F;f)=0$ H-almost everywhere if and only if $F=0$ almost everywhere in $[a,\infty)$.*

Corollary 3. *$k(F;f)$ is algebraically linear in F.*

Proof of Theorem 1 and Corollaries. As $F_{\mathcal{D}}$ and $F_{\mathcal{E}}$ are step functions, so is their difference, equal to L_j, say, in $[b_{j-1},b_j)(1 \leq j \leq n)$ and to 0 in $[b_n,\infty)$. For real valued F,

$$
\begin{aligned}
(17) \qquad 0 \;\leq\; & \{(\mathcal{D})\Sigma' F\Delta f - (\mathcal{E})\Sigma' F\Delta f\}^2 = \left[\sum_{j=1}^{n} L_j\{f(b_j) - f(b_{j-1})\}\right]^2 \\
= \; & \sum_{j=1}^{n} L_j^2\{f(b_j) - f(b_{j-1})\}^2 \\
& + \sum_{j,k=1, j<k} 2L_jL_k\{f(b_j) - f(b_{j-1})\}\{f(b_k) - f(b_{k-1})\}.
\end{aligned}
$$

For $C \equiv (b_1, \ldots, b_n)$ this is a continuous function L over $T(C)$, constant over $T((a,\infty) \setminus C)$. Let R be a generalized interval of T with $C(R) = C$ and $R(b_j)$ compact in one dimension ($j = 1, \ldots, n$). For the Cartesian product I of $I(b)$ with $I \subseteq R$, $C(I) \supseteq C$, let mesh (I) be the greatest lengths of the $I(b)$, omitting $I(b)$ with $b \notin C(I)$. Then, given $\varepsilon > 0$, there is a constant $\delta > 0$ such that if $I \subseteq R$ and mesh $(I) < \delta$, the oscillation of L in I is less then ε. For $\mathcal{P}$ a division of R let mesh $(I) < \delta$ for every generalized interval I with $(I, f) \in \mathcal{P}$ for some f. As $H \geq 0$ the Darboux upper and lower sums for L and $\mathcal{P}$ differ by at most $(\mathcal{P})\Sigma \in H$, and H is VB^* since $H \geq 0$, while by refinement of subdivisions the Darboux lower sums tend to a finite limit. Hence the Riemann and so generalized Riemann integrals of $L(f)H(I)$ exist over R. Expanding the one-dimensional intervals $R(b_j)$ to $(-\infty, \infty)$, the monotone convergence theorem shows that $L(f)H(I)$ is integrable over T. By *Lemma* 3 and (13), (17),

$$
\begin{aligned}
||(\mathcal{D})\Sigma' F\Delta f - (\mathcal{E})\Sigma' F\Delta f||^2 &= \int_T \left[\sum_{j=1}^{n} L_j\{f(b_j) - f(b_{j-1})\}\right]^2 dH \\
= \int_{T(C)} & \left[\sum_{j=1}^{n} L_j^2\{f(b_j) - f(b_{j-1})\}^2 \right. \\
& \left. + \sum_{j,k=1, j<k}^{n} 2L_jL_k\{f(b_j) - f(b_{j-1})\}\{f(b_k) - f(b_{k-1})\}\right] dH.
\end{aligned}
$$

As H is the integral of h in $T(C)$ we can replace H by h and use Fubini's theorem, integrating with respect to $b_n, b_{n-1}, \ldots, b_1$ in turn, b_k before b_j when $j < k$. So we put $x = f(b_k) - f(b_{k-1})$, obtaining in the second sum

$$
\int_{-\infty}^{\infty} (f(b_k) - f(b_{k-1}))r(\ldots, f(b_k) - f(b_{k-1}), \ldots) df(b_k) = \int_{-\infty}^{\infty} x r_k(x; b_k - b_{k-1}) dx = 0.
$$

Thus the integral over T of the terms with $j \neq k$ is 0, leaving the integral of the squares. The integral involving b_j uses $f(b_j)$ as integrator and we have

$$\sum_{j=1}^{n}\int_{-\infty}^{\infty} L_j^2 x^2 r_j(x; b_j - b_{j-1})dx \leq \sum_{j=1}^{n} L_j^2 K(b_j - b_{j-1})$$
$$= K\int_a^{\infty}(F_{\mathcal{D}} - F_{\mathcal{E}})^2 dx = K\|F_{\mathcal{D}} - F_{\mathcal{E}}\|_2^2.$$

For complex valued F take real and imaginary parts separately, adding at the end for (14), and similarly for (15). For (16) let $\mathcal{D}, \mathcal{E}$ be (B, δ)-fine. By Lemma 2, (11), the integral in (14) is

$$\leq K\left\{\|F_{\mathcal{D}} - F\|_2 + \|F - F_{\mathcal{E}}\|_2\right\}^2 < 24K\varepsilon.$$

As $H \geq 0, L^2$-theory gives a suitable sequence of $\mathcal{D}$ for which

$$(\mathcal{D})\Sigma' F\Delta f$$

tends to a limit, say $k(F; f)$, H-almost everywhere. Then Fatou's lemma gives (16) with uniqueness H-almost everywhere.

For Corollary 1 let $p(F; f)$ be the generalized Riemann integral for those f for which it exists, and otherwise let $p(F; f) = k(F; f)$. By Fatou's lemma p satisfies (16), so $p = k$ H-almost everywhere. By (16) for Corollary 2, $k(F; f) = 0$ H-almost everywhere if and only if

$$(\mathcal{D})\Sigma' F\Delta f \to 0, \quad \|F_{\mathcal{D}}\|_2 \to 0$$

with δ, B^{-1}, one way requiring equality in (15) or, more generally, if for some K_1 in $0 < K_1 \leq K$,

$$\|(\mathcal{D})\Sigma' F\Delta f\| \geq \sqrt{K_1}\|F_{\mathcal{D}}\|_2.$$

Then $\|F_{\mathcal{D}}\|_2 \to 0$ with δ, B^{-1}, if and only if $F = 0$ H-almost everywhere.

Corollary 3 is clear.

When H is Wiener measure, T.W. Lee [8] calls $k(F; f)$ *the stochastic integral of F relative to f for the generalized Riemann integral using Wiener measure, in short, the GWS-integral,* and shows that it coincides with the Paley-Wiener-Zygmund [9] integral. Here we use H-measure, so that $k(F; f)$ can be called the *GHS-integral.* Note that if $c > 0$ and $q(x; c) = q(x/\sqrt{c}) > 0$, then (6), (13) imply that H is Gaussian, a normal distribution. See Cramér [1], *p.* 51, *Theorem* 18. Also by Theorem 1,

(18) *for $F \in L^2(a, \infty)$ and measurable $X \subseteq [a, \infty)$, $F\chi(X; \cdot) \in L^2(a, \infty)$ and the GHS-integral exists,*

say as $k(F; f; X)$. We give some properties.

THEOREM 2. *For a sequence (X_n) of disjoint bounded measurable sets in $[a, \infty)$ with union X, then with convergent sum H-almost everywhere,*

$$k(F; f; X) = \sum_{n=1}^{\infty} k(F; f; X_n). \tag{19}$$

In particular, $k(F; f; I)$ is countably additive over disjoint bounded intervals $I \subseteq [a, \infty)$, and so finitely additive over the (I, t) of a division of a finite interval of $[a, \infty)$. For sequences (t_n) of $t, k(F; f; [a, t_n))$ tends to $k(F; f)$ as $t_n \to \infty$, to $k(F; f; [a, u))$ as $t_n \to u-$ $(u > a)$ and $t_n \to u+$ $(u > a)$, and to 0 as $t_n \to a+$. Further, if $F\Delta f\chi(X_n; \cdot)$ is generalized Riemann integrable for $n = 1, 2, \ldots$, then so is $F\Delta\chi(X; \cdot)$. All these results are for H-almost all f.

Proof. In (16) let $F\chi(X_n; \cdot), \varepsilon.2^{-n}, \delta_n$ and $F\chi(X; \cdot), \varepsilon, \delta_0$ replace F, ε, δ, respectively, with $B > a$ if X is unbounded. Let $\delta^+(x) \equiv \min(\delta_0(x), \delta_n(x); x \in X_n,\ n = 1, 2, \ldots)$, $\delta^+(x) = 1$ $(x \notin X)$, and let $\mathcal{D}$ be a (B, δ^+)-fine division of $[a, \infty)$. As $\mathcal{D}$ has only a finite number of (I, t), an integer N depending on $\mathcal{D}$ exists with $n \leq N$ for every $(I, t) \in \mathcal{D}$ with $t \in X_n$. Thus by (16),

$$||(\mathcal{D})\Sigma' f\Delta f\chi(X; \cdot) - k(F; f; X)|| < \varepsilon,$$

$$||(\mathcal{D})\Sigma' f\Delta f\chi(X_n; \cdot) - k(F; f; X_n)|| < \varepsilon.2^{-n},$$

$$\begin{aligned}
||(\mathcal{D})\Sigma' f\Delta f\chi(X; \cdot) &- \sum_{n=1}^{N} k(F; f; X_n)|| \\
&= ||\sum_{n=1}^{N}\{(\mathcal{D})\Sigma' F\Delta f\chi(X_n; \cdot) - k(F; f; X_n)\}|| \\
&\leq \sum_{n=1}^{N} ||(\mathcal{D})\Sigma' F\Delta f\chi(X_n; \cdot) - k(F; f; X_n)|| < \sum_{n=1}^{N} \varepsilon.2^n < \varepsilon,
\end{aligned}$$

$$||\sum_{n=1}^{N} k(F; f; X_n) - k(F; f; X)|| < 2\varepsilon.$$

Here N depends on $\varepsilon > 0$, and goes to infinity as $\varepsilon \to 0+$, so that by Fatou's lemma the series in (19) converges H-almost everywhere and (19) follows. In particular, for H-almost all f, k is countably additive in I (take $X_n = I_n$). For $t_0 = a < t_1, (t_n)$ strictly increasing to infinity, and $I_n = [t_{n-1}, t_n)$ we have $k(F; f; [a, t_n)) \to k(F; f)$, and similarly for the next three results. To prove the generalized Riemann integrability of $F\Delta f\chi(X; \cdot)$ from that of $F\Delta f\chi(X_n; \cdot)$ for each n, we take suitable

$\delta_n(f;x) > 0$ on the disjoint X_n and use (19) and N, so that

$$|(\mathcal{D})\Sigma' F\Delta f\chi(X;\cdot) - k(F;f;X)| \leq \sum_{n=1}^{\infty} |(\mathcal{D})\Sigma' F\Delta f\chi(X_n;\cdot) - k(F;f;X_n)|.$$

Note what we have not proved in the last theorem; for example, see the difference between $t_n \to u+$ and $t \to u+$. To each such (t_n) corresponds a set $Y((t_n))$ of H-variation zero. Given a countable number of such (t_n), the union of the corresponding $Y((t_n))$ is still of H variation zero. But to the uncountable number of t tending to a limit, there corresponds a set Y of f where the limit of $k(F;f;[a,t))$ does not exist, and we cannot prove that Y is of H-variation zero.

Further, given F, let Z be the set of (f,u) for which $k(F;f;[a,t_n))$ does *not* tend to $k(F;f;[a,u))$ (or 0 when $u = a$), for some sequence (t_n) tending to $u+$. Then for each fixed $u \geq a$, the set of f with $(f,u) \in Z$, has H-variation zero. For Lebesgue measure ℓ on $[a,\infty)$, if Z is $H \times \ell$-measurable, then by Fubini's theorem there is a set $W \subseteq T$ of H-variation zero such that if $f \in T \setminus W$ then $(f,u) \in \setminus Z$ for almost all u, and $k(F;f;[a,t_n)) \to k(F;f;[a,u))$ for all (t_n) tending to $u+$ and u almost everywhere in (a,∞). Similarly for $u-$. These results may be as near as we can get to continuity of $k(F;f;[a,t))$ in t.

THEOREM 3. *In Theorem 1 let* (16) *be true for all* (B,δ)*-fine divisions* $\mathcal{D}$ *of* $[a,\infty)$*, with* $\mathcal{P}$ *a partial division of such a* $\mathcal{D}$*. Then the Saks-Henstock Lemma here is that*

(20) $$||(\mathcal{P})\Sigma'(F\Delta f - k(F;f;I)|| \leq \varepsilon.$$

Corollary. $k(F;f;X) = \int_a^\infty \chi(X;\cdot)dk(F;f;I)$ *H-almost everywhere (X measurable).*

Proof of Theorem 3 and Corollary. For $\mathcal{P} \neq \mathcal{D}$ we put $\mathcal{R} = \mathcal{P} \cup \mathcal{E}$, a division of $[a,\infty)$, where $\mathcal{E}$ is any (B,δ)-fine division of the union E of I for all $(I,t) \in \mathcal{D} \setminus \mathcal{P}$. Then, H-almost everywhere,

$$(\mathcal{E})\Sigma' F\Delta f \to k(F;f;E)$$

for a suitable sequence of $\mathcal{E}$, and Fatou's lemma gives

$$||(\mathcal{P})\Sigma' F\Delta f + k(F;f;E) - k(F;f)|| \leq \varepsilon.$$

Finite additivity of k for disjoint intervals gives (20). Multiplying $F\Delta f - k$ by $\chi(X;\cdot)$ is the same as taking partial divisions, and (20), (18) give the Corollary.

[4], *p.* 197, *lines* 5-9, give a short history of the Saks-Henstock lemma, which is [4], *p.* 52, (5.5). The Weierstrass inequality in Burkill integration becomes in generalized Riemann integration the inequality that defines variational integration, see [4], *p.* 53, *Theorem* 5.3 (5.6), and remarks on *p.* 54. The analogue here may be false. For when F is real and $f \in T$ varies, the (I, t) for which $F\Delta f - k \geq 0$, can alter with f, and we fail to prove that

$$||(\mathcal{D})\Sigma'|F\Delta f - k|\,|| \leq 2\varepsilon.$$

However, (20) gives the variation set definition of variational integration.

$k(F; f; [a, t))$ is mean continuous in $t \to u-$, $u > a$, in the following sense.

$$K(t) \equiv ||k(F; f; [t, u))|| \to 0\ (t \to u-, u > a). \tag{21}$$

For if not, then for some $\varepsilon > 0$ and a strictly increasing sequence $(t_n) \to u-$, $K(t_n) \geq \varepsilon$, contrary to (19) when $X_n = [t_{n-1}, t_n)$.

Using (21), and (20) with a single (B, δ)-fine $([v, u), u)$,

$$||F(u))\{f(u) - f(v)\} - k(F; f; [v, u))|| \leq \varepsilon,$$

$$|F(u)|\,||f(u) - f(v)|| \to 0\ (v \to u-, u > a). \tag{22}$$

We can take $F(u) \neq 0$, and also "$u = +\infty$". For finite $u \geq a$, similarly

$$||f(v) - f(u)|| \to 0\ (v \to u+). \tag{23}$$

Now $F_1 \equiv \chi([a, u); \cdot) \in L^2(a, \infty)(a < u < \infty)$. By Theorem 1, (22), (23), and choosing $\delta > 0$ so that some $([v, u), u)$ is in each division of $[a, \infty)$,

$$||f(v) - f(a) - k(F_1; f)|| \to 0\ (v \to u-),\ ||f(u) - f(a) - k(F_1; f)|| = 0,$$

and by uniqueness $k(F_1; f) = f(u) - f(a)$ for H-almost all f. By linearity in F and Theorem 2,

$$\begin{aligned} k(\chi([u, v); \cdot); f) &= f(v) - f(u), \\ k(\chi((u, v); \cdot); f) &= f(v) - f(u+)(a \leq u < v), \end{aligned} \tag{24}$$

for H-almost all f. Theorem 2 then enables us to tackle open sets with a convergent infinite series, and so closed sets, for H-almost all f.

If $f \in L^2(a, \infty)$ is also of bounded variation, then for continuous f, (1) can be a Riemann-Stieltjes integral and so a generalized Riemann integral. But an examination of this encounters a result of Doob for Wiener measure, in that the set of continuous functions f forms a non-Wiener-measurable set.

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