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Upper and Lower Bounds for the Packing Measure in Relation to the Hausdorff Measure

In [3], Tricot and Raymond show that if the Hausdorff measure of a set is equal to the packing measure of the set (assuming that the Hausdorff dimension of the set is equal to the packing dimension of the set), then the dimension is an integer. In this paper, the author considers the case when the Hausdorff measure is not equal to the packing measure. Since the packing measure is the largest dimensional measure, it would be informative to have an estimate of how much larger the packing measure is with respect to the Hausdorff measure. Also, it would be informative to have an upper bound for the packing measure. It is assumed in this paper that information about the Hausdorff measure on a set is known so that a Hausdorff density function and be used to calculate an upper bound and a lower bound for the packing measure on that set. The first part of the paper is a generalization of an inequality in [3], and the second part of the paper considers the special case of the symmetric (Cantor-like) set on the real line.

Let the function $h : [0, \infty] \to R^+$ be continuous, increasing, $h(0) = 0$, and $\limsup_{r\to 0} h(2r)/h(r) = h^*$ be finite. An example of such a function is $h(x) = x^\alpha$, $\alpha > 0$, where $h^* = 2^\alpha$.

First, the following definitions are needed. In these definitions, $B(x, r)$ is the ball with center x and radius r in R^n, and $u^h(E)$ will denote the Hausdorff measure of a set E.

<u>Definition 1</u>: (Packing measure). The premeasure of a set E is defined by $H_p(E) = \lim_{\delta\to 0}[\sup\{\sum_i h(2r_i)$: the balls $B(x_i, r_i)$ are pairwise disjoint, $x_i \in E$, and $r_i < \delta\}]$. The packing measure is $h_p(E) = \inf\{\sum_i H_p(E_i) : E \subset U_i E_i\}$.

<u>Definition 2</u>: (Symmetric Derivation Basis Measure). Let $\delta(x)$ be a positive real function on R^n. Then $H_s(E) = \sup\{\sum_i h(2r_i)$: the collection of $B(x_i, r_i)$ are pairwise disjoint, $x_i \in E$, and $r_i < \delta(x_i)\}$. The symmetric derivation basis measure is $h_s(E) = \inf\{H_s(E) : \delta(x)$ is any positive function$\}$.

It was shown in [1] that the packing measure is the symmetric derivation basis measure.

Definition 3: Lower Hausdorff density function. $\underline{d}_m^h(x) = \liminf_{r\to 0} u^h[E \cap B(x,r)]/h(2r)$.

A lower bound for the packing measure is given in the first theorem.

Theorem 1: If $\{E_n\}$ are disjoint, h_p-measurable sets such that $E = U_n E_n$,and if $\underline{d}_m^h(x) < a_n < 1$ for $x \in E_n$, a.e. u^h, then $h_p(E) \geq \sum_n a_n^{-1} u^h(E_n)$.

Proof: Since $\underline{d}_m^h(x) < a_n < 1$, for each $\delta(x) > 0$, there exist infinitely many $r < \delta(x)$ such that $u^h[E_n \cap B(x,r)]/h(2r) < a_n$. Therefore, $a_n^{-1} u^h[E_n \cap B(x,r)] < h(2r)$. Using the Vitali covering theorem for Hausdorff measures and the fact that the symmetric derivation basis measure is the packing measure, $a_n^{-1} u^h(E_n) \leq h_p(E_n)$. Since the sets $\{E_n\}$ are h_p-measurable, $\sum_n a_n^{-1} u^h(E_n) \leq h_p(E)$.

The obvious corollary is the following:

Corollary 1: If $\underline{d}_m^h(x) < a < 1$, a.e. u^h on E, then $a^{-1} u^h(E) \leq h_p(E)$.

Let $h(x)$ be defined as above and let the function $g : [0, \infty) \to R^+$ be continuous, increasing, $g(0) = 0$. Suppose further that $\limsup_{r\to 0} g(2r)/g(r) = g^*$ is finite, and $g(x) \leq h(x)$ for all $x < 1$. An example would be $g(x) = x^\beta$ and $h(x) = x^\alpha$, when $\alpha \leq \beta$.

Theorem 2: Let $g(x) \leq h(x)$ for all $x < 1$ and let $\{E_n\}$ be pairwise disjoint sets such that $E = U_n E_n$. If $0 < a_n < \underline{d}_m^h(x)$ for $x \in E_n$, a.e. g_p, if $g_p(E) < \infty$, and if $\sum_n a_n^{-1} u^h(E_n) < \infty$, then $g_p(E) \leq \sum_n a_n^{-1} u^h(E_n)$.

Proof: Since $0 < a_n < \underline{d}_m^h(x)$ for $x \in E_n$, there exists an $r_0(x) < 1/2$ such that if $r < r_0(x)$, then $a_n < u^h[E_n \cap B(x,r)]/h(2r)$. Therefore, $h(2r) < a_n^{-1} u^h[E_n \cap B(x,r)]$. Since $r_0(x) < 1/2$, $2r < 1$ and $g(2r) < a_n^{-1} u^h[E_n \cap B(x,r)]$. Therefore, for any positive function $\delta(x) < r_0(x)$ for all $x \in E_n$, $G_s(E_n) \leq a_n^{-1} u^h(E_n)$. Hence, $g_s(E_n) \leq a_n^{-1} u^h(E_n)$ and $g_p(E) \leq \sum_n g_p(E_n) \leq \sum_n a_n^{-1} u^h(E_n)$.

The obvious corollary is the following:

Corollary 2: Let $g(x) \leq h(x)$ for all $x < 1$. If $0 < a < \underline{d}_m^h(x)$ for $x \in E$, a.e. g_p and $g_p(E) < \infty$, then $g_p(E) \leq a^{-1} u^h(E)$.

Theorem 2 and Corollary 2 are of special interest when $g(x) = x^\beta$ and $h(x) = x^\alpha$. In this case, β is the packing measure dimension of a set E, $\beta = \mathrm{Dim}\, E$, and α is the Hausdorff dimension of the set E, $\alpha = \dim E$. The inequality $\alpha = \dim E \leq \mathrm{Dim}\, E = \beta$ always holds.

2 - The Symmetric Set

For a specific class of sets on the real line, the lower bound for the packing measure is studied. This class of sets is given in the following definition.

<u>Definition 5</u>: (Symmetric Set). For $n = 1$, remove an interval of length b_1 from the center of $[0,1]$ leaving two closed intervals each of length a_1. For $n = k$, remove an interval of length b_{k+1} from the center of each of the 2^k closed intervals of length a_k (with $b_{k+1} < a_k$) leaving 2^{k+1} closed intervals of length a_{k+1}. The symmetric set, E, is defined as $E = \cap_n[U_{i=1}^{2^n} A_n{}^i]$ where $\{A_n{}^i\}_{i=1}^{2^n}$ are the closed intervals of length a_n.

For this section $h(x) = x^\alpha$, where $0 < \alpha < 1$, and the following notation is used: $h_p = p^\alpha$, $u^h = u^\alpha$ and $h_s = s^\alpha$. Also, assume α is the packing measure dimension of the symmetric set E and the Hausdorff measure dimension of E. It is given in [2] that when the dimensions are equal on the symmetric set E, then $\alpha = \lim_{n\to\infty}[\log 2^{-n}/\log a_n]$.

An elementary computation now shows that:

<u>Observation 1</u>: Let E be a symmetric set. Then $\alpha = \lim_{n\to\infty}[\log 2^{-n}/\log a_n]$ if and only if $\lim_{n\to\infty} \sqrt[n]{a_n} = (1/2)^{1/\alpha}$. As $\liminf a_{n+1}/a_n \leq \lim a_n{}^{1/n}$ it follows that $\liminf a_{n+1}/a_n \leq 2^{-1/\alpha}$.

The following theorem shows that under certain conditions a constant $\gamma < 1$ exists such that $\underline{d}_m{}^\alpha(x,E) \leq \gamma$ for all x in the symmetric set E.

<u>Theorem 3</u>: Let E be a symmetric set. If the Hausdorff measure dimension α is equal to the packing measure dimension, then $d_m{}^\alpha(x,E) \leq (1/2)^\alpha[1/(2^{1/\alpha}-2)]^\alpha$.

<u>Proof</u>: Since $\liminf_{n\to\infty}(a_{n+1}/a_n) \leq (1/2)^{1/\alpha} < \beta < (1/2)$, there exist infinitely many n such that $(a_{n+1}/a_n) < \beta$. Hence, for those same infinitely many (a_{n+1}, a_n), $b_{n+1}/a_n = (a_n - 2a_{n+1})/a_n > (1-2\beta)$. Therefore, $a_n/b_{n+1} < 1/(1-2\beta)$. Now, let x be any endpoint of E, and let b_m be the length of the contiguous interval of E with one of the endpoints being x. Then, there exist infinitely many of the above (a_{n+1}, a_n) such that $b_{n+1} + a_{n+1} < b_m$. Let $r = a_{n+1} + b_{n+1}$. Therefore

$$u^\alpha[E \cap B(x,r)]/(2r)^\alpha < (a_{n+1})^\alpha/(2b_{n+1})^\alpha =$$
$$(1/2)^\alpha(a_n/b_{n+1})^\alpha(a_{n+1}/a_n)^\alpha < (1/2)^\alpha[1/(1-2\beta)]^\alpha[\beta]^\alpha.$$

Let x be a limit point of E which is not an endpoint of E. Then x is contained in infinitely many closed intervals of length (a_{n+1}, a_n) given above with respect to the endpoints. Therefore, $r \geq b_{n+1}$ and

$$u^\alpha[E \cap B(x,r)]/(2r)^\alpha \leq (a_{n+1})^\alpha/(2b_{n+1})^\alpha < (1/2)^\alpha[1/(1-2\beta)]^\alpha[\beta]^\alpha.$$

Letting $n \to \infty$ and β approach $(1/2)^{1/\alpha}$,

$$\underline{d_m}^\alpha(x,E) \leq (1/2)^\alpha[(2^{1/\alpha}/(2^{1/\alpha}-2)]^\alpha(1/2) = (1/2)^\alpha[1/(2^{1/\alpha}-2)]^\alpha.$$

The following Corollary is immediate.

Corollary 3: Let E be a symmetric set. If the Hausdorff dimension α is equal to the packing dimension and if $(1/2)^\alpha[1/(2^{1/\alpha}-2)]^\alpha < 1$, then $p^\alpha(E) \geq 2^\alpha(2^{1/\alpha}-2)^\alpha u^\alpha(E)$.

It can be observed that $2 = 2^\alpha(2^{1/\alpha}-2)^\alpha$ when $\alpha = 1/2$. Therefore $p^\alpha(E) \geq 2u^\alpha(E)$ when $\alpha < 1/2$. It can also be observed that α must be less than $\log 2/\log(5/2)$ in order for Corollary 3 to be valid and this happens when α is less than or equal to approximately 0.75647.

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