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Lusin's Theorem

1. Lusin's theorem [1], [2], is one of the simplest important theorems in classical real analysis. Its character is that of a folk theorem and seems to have been known to the Italian mathematicians even before it was published by Lusin.

We state the theorem for measurable functions on the interval $I = [0,1]$. Let $f : I \to R$ be a measurable function. For every $\varepsilon > 0$ there is a closed set $F \subset I$ with $m(F) > 1 - \varepsilon$, such that f is continuous on F relative to F. By the Tietze extension theorem, we may state Lusin's theorem in the following form. If $f : I \to R$ is measurable then for every $\varepsilon > 0$ there is a continuous $g : I \to R$ such that $f(x) = g(x)$, except on a set of measure less than ε. It seems plausible that there is a Baire 1 function equal to f almost everywhere. It is interesting that this is not true. However, it is true that for every measurable
$f : I \to R$ there is a Baire 2 function g such that $f(x) = g(x)$ almost everywhere.

It is convenient for us to express Lusin's theorem in another form. Denjoy defined approximate continuity and showed that $f : I \to R$ is measurable if and only if it is approximately continuous almost everywhere. Then Lusin's theorem takes the following form.

A function $f : I \to R$ is such that for every $\varepsilon > 0$ there is a continuous g such that the set of x for which $f(x) \neq g(x)$ is of measure less than ε if and only if f is approximately continuous almost everywhere.

2. We noted that for every measurable $f : I \to R$ there is a Baire 2 function g such that $f(x) = g(x)$ almost everywhere. We now state a number of related facts.

The above function of Baire class 2 may be taken to be the limit of an increasing sequence of lower semicontinuous functions. This is known as the Vitali-Caratheodory theorem.

The following question has been considered. For a measurable $f : I \to R$ is there a homeomorphism of I onto I such that $f \circ h$ is almost everywhere equal to a Baire 1 function? Gorman, [3], showed that if f is measurable and assumes only finitely many values then there is a homeomorphism h such that $f \circ h$ is almost everywhere equal to a Baire 1 function. He then showed that this is false in general. He gave an example of a measurable f such that $f \circ h$ is not equivalent to a Baire 1 function for any homeomorphism h. It was later observed that the function f of this example, although measurable, is such that, for some homeomorphism h, the function $f \circ h$ is not measurable. In this regard, a function f has been called absolutely measurable if $f \circ h$ is measurable for every homeomorphism h. It was then shown by Bruckner, Davies, and Goffman, [4], that if f is absolutely measurable there is a homeomorphism h such that $f \circ h$ is almost everywhere equal to a Baire 1 function. The proof is not easy.

3. It is natural to ask about arbitrary, rather than measurable, functions. Sierpinski [5] proved the existence of an f such that, for every E of the power of the continuum, f is not continuous on E relative to E. Using the continuum hypothesis, f is not continuous on any uncountable E relative to E. The analogue of Lusin's theorem, if there is one, must be far weaker for arbitrary functions. A noteworthy theorem of Blumberg, [6], assumes this role. The theorem asserts that for any arbitrary $f : I \to R$ there is an everywhere dense set $E \subset I$ such that f is continuous on E relative to E. By the Sierpinski example, E may have to be countable. A considerable amount of work has been done on spaces for which Blumberg's theorem holds. A topological space is called a Blumberg space if for every $f : X \to R$ there is an everywhere dense $E \subset X$ such that f is continuous on E relative to E. Motivation for this work was furnished by a simple result of Bradford and Goffman, [7]. A topological space X is called a Baire space if every open set in X is of the second category of Baire, i.e., it is not the union of countably many nowhere dense sets. The theorem in question says that a metric space is a Blumberg space if and only if it is a Baire space. Of the considerable amount of research which followed, only one result will be mentioned. The density topology on R has as its open sets those measurable sets $E \subset R$ such that every $x \in E$ is a point of metric density 1 of E. It is a fact, not hard to prove, that this is a topology. H.E. White, [8], showed that the reals with the density topology constitute a Baire space which is not a Blumberg space.

There is a subtle and beautiful theorem on arbitrary functions which serves as an analogue of the theorem that every measurable function is equal almost everywhere to a Baire 2 function. This is a theorem of Saks and Sierpinski, [9], who proved

the following noteworthy fact in 1928. If $f : I \to R$ is arbitrary, there is a function $g : I \to R$ of Baire type 2 such that, for every $\varepsilon > 0$, $|f(x) - g(x)| < \varepsilon$ on a set of outer measure greater than $1 - \varepsilon$. A proof was given by Blumberg, [10], in 1935 using his measurable

boundaries of an arbitrary function. The measurable boundaries of an arbitrary real function f are measurable functions u and ℓ such that, for almost every x, $\ell(x) \leq f(x) \leq u(x)$, and both $u(x)$ and $\ell(x)$ are fully approached by f, i.e., for every $\varepsilon > 0$, and almost every ξ, the sets for which $|f(x) - u(x)| < \varepsilon$ and $|\ell(x) - f(x)| < \varepsilon$ have outer metric density 1 at ξ. The Saks-Sierpinski theorem follows readily.

A simple proof was given in 1948 by Goffman, [11]. The proof uses the fact, not hard to prove, that for every $\varepsilon > 0$ there is a continuous g such that $|f(x) - g(x)| < \varepsilon$ on a set of outer measure greater than $1 - \varepsilon$, and the following lemma. For every $\varepsilon > 0$, $\eta > 0$, and continuous g such that $|f(x) - g(x)| < \varepsilon$ on a set of outer measure greater than $1 - \varepsilon$, there is a continuous h such that $|g(x) - h(x)| < \varepsilon$ on a set of measure greater than $1 - \varepsilon$, and $|f(x) - h(x)| < \eta$ on a set of outer measure greater than $1 - \eta$. These matters have been treated in a general setting by Goffman and Zink [12].

4. The functions, f, which are such that for every $\varepsilon > 0$ there is a continuous g such that $f(x) = g(x)$, except on a set of measure less than ε, are the functions which are approximately continuous almost everywhere. What about the functions which may be approximated in this way by the continuously differentiable functions? This question was treated by Hassler Whitney, [13]. We first note that Lusin's theorem holds for functins on R^n with little change in the proof. We recall that a function f on R^n is approximately differentiable at a point ξ if its total differential exists at ξ relative to a set E whose metric density is 1 at ξ. Whitney showed that f is such that for every $\varepsilon > 0$ there is a $g \in C^1$ such that the set at which $f(x) \neq g(x)$ has measure less than ε if and only if f is approximately differentiable almost everywhere. The proof is easy for $n = 1$ but is difficult for $n > 1$ and uses the famous Whitney extension theorem.

5. We now turn to Lusin type theorems for one-one mappings. We first discuss the case of arbitrary one-one mappings. Let $S \subset R^2$ be an arbitrary set which, for convenience, has outer measure 1. Let $F = (f, f^{-1})$ and $G = (g, g^{-1})$ be arbitrary one-one mappings of S onto itself. Let $|p - q|$ be the distance between points p and q in R^2. The distance $d(F, G)$ between F and G is defined as the infimum of all $\varepsilon > 0$ for which there are sets A and B in S with $m_e(A) > 1-\varepsilon$ and $m_e(B) > 1-\varepsilon$, such that for every $x \in A$, $|f(x)-g(x)| < \varepsilon$ and for every $x \in B$, $|f^{-1}(x) - g^{-1}(x)| < \varepsilon$. Goffman obtained the following result, [14], in 1943.

THEOREM. *If Q is a closed unit square and $F = (f, f^{-1})$ is a one-one mapping of Q onto itself, then for every $\varepsilon > 0$ there is a homeomorphism $G = (g, g^{-1})$ of Q itself such that $d(F, G) < \varepsilon$.*

This theorem holds for all $n \geq 2$ but fails to hold for $n = 1$.

The measurable case is more interesting, [15]. A one-one mapping $F = (f, f^{-1})$ is measurable if f and f^{-1} are both measurable. Let I_n be the closed unit n cube, $n \geq 2$, and let $F = (f, f^{-1})$ be one-one and measurable on I_n onto I_n. Then, for every $\varepsilon > 0$, there is a homeomorphism $G = (g, g^{-1})$ of I_n onto itself such that $f(x) = g(x)$ and $f^{-1}(x) = g^{-1}(x)$ on sets of measure greater than $1 - \varepsilon$. This fact does not hold for $n = 1$ as the following example shows. Let

$$f(x) \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 1 - 2x, & \frac{1}{2} < x \leq 1. \end{cases}$$

There is no homeomorphism which has the needed property for $\varepsilon = \frac{1}{4}$. For $n = 2$, the proof may be accomplished by first showing that, for any $\varepsilon > 0$, F is a homeomorphism between zero dimensional compact sets of measure greater than $1 - \varepsilon$ and then extending this homeomorphism to one between I_2 and itself. This proof does not work for $n \geq 3$. Indeed, Antoine, [16], gave an example of a homeomorphism between zero dimensional compact sets in I_3 which does not extend to one between I_3 and itself.

In this regard we introduced in [15], a notion of sectionally zero dimensional sets. A set $E \subset I_n$ is called sectionally zero dimensional if for every hyperplane π, parallel to a face of I_n, and $\varepsilon > 0$, there is a hyperplane π' parallel to π, whose distance from π is less than ε, which contains no points of E. It is shown that for every measurable $F = (f, f^{-1})$ there are compact, sectionally zero dimensional sets A and B in I_n each of measure greater than $1 - \varepsilon$, such that F is a homeomorphism between A and B. It is then shown that every homeomorphism between compact sectionally zero dimensional sets in I_n can be extended to a homeomorphism between I_n and itself, for $n > 1$. These facts yield the desired result.

The following related fact holds for the case $1 \leq n < m$. If $1 \leq n < m$ and $F = (f, f^{-1})$ is a one-one measurable mapping of I_n onto I_m then, for every $\varepsilon > 0$, there is a homeomorphism $G = (g, g^{-1})$ between I_n and a subset of I_m such that $f(x) = g(x)$ on a set of n measure greater than $1 - \varepsilon$ and $f^{-1}(x) = g^{-1}(x)$ on a set of m measure greater than $1 - \varepsilon$.

6. There are results which bear a relation to Whitney's theorem similar to the one the above result bears to Lusin's theorem. This is the work of H.E. White, [17]. Some definitions are needed.

A one-one mapping (T, T^{-1}) of I_n onto itself is called a C^1 diffeomorphism if T and T^{-1} are both continuously differentiable. The problem is to find necessary and sufficient conditions on a one-one measurable transformation (T, T^{-1}) in order that, for every $\varepsilon > 0$, there be a diffeomorphism (S, S^{-1}) such that $T(x) = S(x)$ and $T^{-1}(x) = S^{-1}(x)$ on sets of measure greater than $1 - \varepsilon$. It is immediate that a necessary condition is that T and T^{-1} be approximately differentiable almost everywhere. Two other conditions which appear to be necessary from the fact that they are obeyed by diffeomorphisms are a) both T and T^{-1} take sets of measure zero into sets of measure zero and b) the Jacobian has the same sign almost everywhere. These three conditions turn out to be sufficient. This yields the

THEOREM. *A one-one measurable transformation (T, T^{-1}) of I_n onto itself, $n > 1$, is such that for every $\varepsilon > 0$ there is a C^1 diffeomorphism (S, S^{-1}) such that $T(x) = S(x)$ and $T^{-1}(S) = S^{-1}(x)$ on sets of measure greater than $1 - \varepsilon$ if and only if T and T^{-1} are almost everywhere approximately differentiable, the approximate Jacobian of T is almost everywhere greater than or equal to zero, or almost everywhere less than or equal to zero, as is the approximate Jacobian of T^{-1}, and T and T^{-1} take sets of measure zero into sets of measure zero.*

7. The question regarding one-one mappings has been considered for Blumberg's theorem as well. Let $I = [0, 1]$ and let (f, f^{-1}) be an arbitrary one-one mapping of I onto I. It is asked whether there are everywhere dense sets A and B in I such that (f, f^{-1}) is a homeomorphism between A and B. In this regard, Goffman, [18], gives an example of an (f, f^{-1}) of I onto itself such that if $A \subset I$ is everywhere dense and if f is continuous on A relative to A then either $f(A)$ is not everywhere dense or there is an $x \in f(A)$ such that f^{-1} is not continuous at x relative to $f(A)$. This shows that the sought result is false.

8. Lusin type theorems exist for three important classes of functions. For $n = 1$, these are the absolutely continuous functions, AC, the functions of bounded variation, BV, and the continuous functions of bounded variation, CBV. Let the domain be $I = [0, 1]$. For $f \in BV$ the following fact holds, [19].

If $f \in BV$, for every $\varepsilon > 0$ there is a $g \in C^1$ such that $f(x) = g(x)$ on a set of measure greater than $1 - \varepsilon$ and $|v_f - v_g| < \varepsilon$. The notation v_f designates the variation of f on I. For absolutely continuous functions the following holds. A function of bounded variation is absolutely continuous if and only if for every $\varepsilon > 0$ there is a $g \in C^1$ such that if E is the set for which $f(x) \neq g(x)$ then $m(E) < \varepsilon$, $\int_E |f'(x)|dx < \varepsilon$, and $\int_E |g'(x)|dx < \varepsilon$.

This fact may be expressed in terms of distribution derivative. For a function f of bounded variation, the derivative, in the sense of distributions, is a totally finite measure. This measure is the integral of a summable function if and only if the function f is absolutely continuous, and is a nonatomic measure if and only if the function f is continuous and of bounded variation. Accordingly, f is absolutely continuous if and only if for every $\varepsilon > 0$, there is a $g \in C^1$ such that if E is the set for which $f(x) \neq g(x)$ then $m(E) < \varepsilon$, $v_f(E) < \varepsilon$ and $v_g(E) < \varepsilon$.

The case for functions in CBV is much more subtle. The interesting treatment of this topic is the work of Fon-che Liu, [20]. Let $[-\infty, \infty]$ be the two point compactification of the real line.

An extended real valued function on $I = [0, 1]$ is called weakly continuous on I if it is continuous on I into $[-\infty, \infty]$. A real valued function on I is called a weak C^1 function if its derivative exists (possibly $+\infty$ or $-\infty$) and is weakly continuous. The following theorem holds. If f is real valued, continuous, and of bounded variation on $I = [0, 1]$, for every $\varepsilon > 0$ there is a weak C^1 function g such that, if E is the set for which $f(x) \neq g(x)$, then $m(E) < \varepsilon$, $v_f(E) < \varepsilon$, and $v_g(E) < \varepsilon$.

The proof is delicate and uses the following decomposition theorem which is of independent interest.

For $f : I \to R$ of type CBV, there is a decomposition of I into sets

$$I = (\cup_{n=1}^{\infty} A_n) \cup (\cup_{n=1}^{\infty} B_n) \cup N$$

which are pairwise disjoint, f is monotonically nondecreasing on each A_n, monotonically nonincreasing on each B_n and $v_f(N) = 0$.

9. There are Lusin type theorems for classes of functions of n variables which are analogous to the classes AC, BV, and CBV of functions of one variable, [21], [22], [23]. These classes will now be defined. Let $(x_1, \ldots, x_n)$ be a rectangular coordinate system in n space. Let I_n be the unit n cube. A function $f : I_n \to R$ is said to be of type BV if there is an equivalent g such that, for every $i = 1, \ldots, n$, g is of bounded variation in x_i for all values of the remaining $n - 1$ variables, and these variation functions are all summable. If the equivalent function g may be taken to be absolutely continuous in x as well, then f is said to be in AC. Finally, if instead of absolute continuity, we have continuity, then f is said to be linearly continuous. These definitions may be phrased in terms of distributions. The functions in BV are those whose distribution partial derivatives are totally finite measures. For $n = 1$, all such measures are obtained in this way. For $n > 1$, those measures which are absolutely continuous with respect to Hausdorff $n - 1$ measure are obtained. The functions in AC are those whose partial derivatives are absolutely continuous with respect to Lebesgue n measure. The linear continuous functions in BV are those whose partial derivatives are measures which vanish on sets of finite Hausdorff $n-1$ measure. The apparent difference between $n = 1$ and $n > 1$ is discussed in [24] and is shown to be unreal.

Lusin type theorems will now be stated for these classes of functions. Lebesgue n measure will be designated by $m(E)$, the n partial derivatives by $v_1(E), \ldots, v_n(E)$, and the total variation measure by $v(E)$. Recall that the total variation measure is defined as follows. Let $f : I_n \to R$ be in BV and let $v_1, \ldots, v_n$ be the partial derivative measures of f. For every Borel set E, let

$$v(E) = \sup \sum_{j=1}^{k} [\sum_{i=1}^{n} (v_i(E))^2]^{1/2}$$

for all finite partitions $E_1, \ldots, E_k$ of E into pairwise disjoint Borel sets. The following Lusin type theorems hold for BV and AC. Let $f : I_n \to R$ be in BV and let $v_{i,f}$ be the partial derivative measures and v_f the total variation measure of f. For every $\varepsilon > 0$, there is a $g \in C^1$ such that $f(x) = g(x)$, except on a set of measure less than ε, and $|v_f(I_n) - v_g(I_n)| < \varepsilon$. This interesting theorem was proved by J.H. Michael, [25], with g Lipschitzian . The slight improvement from Lipschitzian to C^1 was made by Goffman, [26], who also showed that this result is best possible. For $f \in AC$, the Lusin type theorem is as follows. If $f : I_n \to R$ is in AC then, for every $\varepsilon > 0$, there is a $g \in C^1$ such that if E is the set for which $f(x) \neq g(x)$, then $m(E) < \varepsilon$, $v_f(E) < \varepsilon$, and $v_g(E) < \varepsilon$. The proof follows from the BV case.

The situation is more complicated for the linearly continuous functions of bounded variation, [27]. Designate this class by L. It is noted that functions in L may be everywhere discontinuous. The cases $n = 2$ and $n > 2$ are treated separately. The latter seems to be much more difficult.

Let $n = 2$. Let $f : I_2 \to R$ be in L, i.e., linearly continuous and of bounded variation. The Lusin type theorem is as follows. For every $\varepsilon > 0$, there is an everywhere continuous $g \in BV$ such that if E is the set for which $f(x) \neq g(x)$, then $m(E) < \varepsilon$, $v_f(E) < \varepsilon$ and $v_g(E) < \varepsilon$, [28].

For $n > 2$, the known result is as follows. If $f \in L$, then for every $\varepsilon > 0$ there is an everywhere approximately continuous $g \in BV$ such that if E is the set for which $f(x) \neq g(x)$ then $m(E) < \varepsilon$, $v_f(E) < \varepsilon$, and $v_g(E) < \varepsilon$. The converse also holds, [29].

An interesting side remark is that if $f \in BV$ is everywhere approximately continuous then $f \in L$.

11. Finally, Lusin type theorems hold for Sobolev spaces. Let U_n be the unit ball in n space and let A be the functions with compact support in U_n which are of class C^1.

Suppose $n > 1$. For every $p \geq 1$ consider the norm

$$\|f\|_1^p = |f|_p + \sum_{i=1}^{n} |\frac{\partial f}{\partial x_i}|^p$$

where $|\cdot|_p$ is the usual L_p norm.

The completion W_1^p of A_1 with this norm is a Banach space which is called a Sobolev space.

F.C. Liu obtained the following important result, [30]. If $f \in W_p^1$, then for every $\varepsilon > 0$, there is a $g \in C^1$ such that if E is the set for which $f(x) \neq g(x)$ then the Sobolev norms of f and g, restricted to E, are less than ε.

Similar results were obtained by Liu for higher order Sobolev spaces. Deep refinements of these results were obtained by Michael and Ziemer, [31].

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