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A MINIMAL FAMILY OF OPEN INTERVALS GENERATING THE BOREL SETS

Let $\mathcal{F}$ be the family of all open intervals of R, and let $\mathcal{B}_R$ denote the Borel sets of R. The following two statements appear in [2, p. 19]:

> "A subfamily $\mathcal{F}_0 \subset \mathcal{F}$ is a generator for $\mathcal{B}_R$ iff the set of end points of intervals in $\mathcal{F}_0$ is dense in R. Thus if $\mathcal{F}_0 \subset \mathcal{F}$ is a generator for $\mathcal{B}_R$ then by removing any finitely many intervals from $\mathcal{F}_0$ we still get a generator for $\mathcal{B}_R$."

Małgorzata Filipczak [1] has shown that the first statement is false. We show that the second statement is also false by making use of the fact that if a σ-algebra separates points, then so does a generator for that σ-algebra [1, Lemma 1]. Since R is homeomorphic to the open interval $(0,1)$ and since homeomorphism preserves open intervals[1], it suffices to give examples in $(0,1)$. More precisely, we find a minimal family $\mathcal{E}$ of open intervals in $(0,1)$ such that $\mathcal{B}_{(0,1)}$ is the smallest σ-algebra containing $\mathcal{E}$.

Example 1. For each positive integer n, let

$$\mathcal{E}_n = \{((k-1)\cdot 2^{-n}, k\cdot 2^{-n}) : k = 1\ldots 2^n\},$$

and let $\mathcal{E} = \bigcup \mathcal{E}_n$. The open intervals in $\mathcal{E}_1$ will be called members of the first level, those in $\mathcal{E}_2$ members of the second level, etc.

[1]In this paper an open interval is an interval that happens to be an open set. Example 2 is given for those who want open intervals to have compact closure.

The σ-algebra $\mathcal{S}$ generated by $\mathcal{E}$ is contained in $\mathcal{B}_{(0,1)}$. We check that $\mathcal{B}_{(0,1)} \subset \mathcal{S}$. Let

$$A = \{k \cdot 2^{-n} : k \text{ and } n \text{ are positive integers and } k < 2^n\},$$

and suppose $x \in A$, where $x = (2l-1) \cdot 2^{-n}$ in reduced form. If $n \geq 2$, then

$$\{x\} = ((l-1) \cdot 2^{-n+1}, l \cdot 2^{-n+1}) \setminus [((2l-2) \cdot 2^{-n}, (2l-1) \cdot 2^{-n}) \cup ((2l-1) \cdot 2^{-n}, 2l \cdot 2^{-n})].$$

But if $n = 1$ (so that $x = 2^{-1}$), then

$$\{x\} = (0,1) \setminus [(0, 2^{-1}) \cup (2^{-1}, 1)].$$

Hence, every subset of A belongs to $\mathcal{S}$. If U is any open set in $(0,1)$, then U can be expressed as the union of all $E \in \mathcal{E}$ contained in U along with some subset of A. Hence, $\mathcal{E}$ generates $\mathcal{B}_{(0,1)}$.

Since $\mathcal{B}_{(0,1)}$ separates points in $(0,1)$, so must any generator. Let us check that if some member of $\mathcal{E}$ is removed, then the resulting family no longer separates each pair of points in $(0,1)$. Suppose $E = ((k-1) \cdot 2^{-n}, k \cdot 2^{-n})$, where $k \in \{1, \ldots, 2^n\}$. Let x be the left endpoint $(k-1) \cdot 2^{-n}$ if k is even, and let x be the right endpoint $k \cdot 2^{-n}$ if k is odd. Clearly E separates its midpoint $y = (2k-1) \cdot 2^{-n-1}$ from x. Neither x nor y belongs to any other member of $\mathcal{E}_n$. If $m > n$, then neither x nor y belongs to any member of $\mathcal{E}_m$. Notice that x is not an endpoint for any previous level. Thus if $m < n$, then any member of $\mathcal{E}_m$ which contains one of x or y must contain both. Hence, $\mathcal{E}$ is a minimal generator for $\mathcal{B}_{(0,1)}$.

Example 2. An easy modification of Example 1 yields a minimal generating family of open intervals having compact closure. For the family $\mathcal{E}$ of Example 1, remove all open intervals whose closure is not compact. Then, for $n \geq 3$, replace the intervals $(2 \cdot 2^{-n}, 3 \cdot 2^{-n})$ and $(1 - 3 \cdot 2^{-n}, 1 - 2 \cdot 2^{-n})$ by $(2^{-n}, 3 \cdot 2^{-n})$ and $(1 - 3 \cdot 2^{-n}, 1 - 2^{-n})$, respectively, to get a new family $\mathcal{D}$ of open intervals with compact closure. Once again, let us show that if $x \in A$ then $\{x\}$ is in the σ-algebra $\mathcal{S}$ generated by $\mathcal{D}$. If $x = (2l-1)\cdot 2^{-n}$ in reduced form and if l is neither 1 nor 2^{n-1}, then

$$\{x\} = ((l-1)\cdot 2^{-n+1}, l\cdot 2^{-n+1}) \setminus [((2l-2)\cdot 2^{-n}, (2l-1)\cdot 2^{-n}) \cup ((2l-1)\cdot 2^{-n}, 2l\cdot 2^{-n})].$$

We now consider the case when l is 1 or 2^{n-1}, so that x has the form 2^{-n} or $1-2^{-n}$ for some n. For each n, we have

$$(0, 2^{-n}) = (2^{-n-1}, 2^{-n}) \cup \bigcup_{m \geq n+2} (2^{-m}, 3\cdot 2^{-m}),$$

so that $(0,2^{-n}) \in \mathcal{S}$. Similarly, each $(1-2^{-n}, 1) \in \mathcal{S}$. Evidently, $\{1/2\} \in \mathcal{S}$ since $\{1/2\} = (0,1) \setminus [(0,1/2) \cup (1/2,1)]$. If $n \geq 2$, then $\{2^{-n}\} = (0, 2^{-n+1}) \setminus [(0,2^{-n}) \cup (2^{-n}, 2^{-n+1})]$, so that $\{2^{-n}\} \in \mathcal{S}$. Similarly, $\{1-2^{-n}\} \in \mathcal{S}$. Thus every subset of A lies in $\mathcal{S}$ and every open set lies in $\mathcal{S}$ by the same reasoning as in Example 1.

It remains to show that each member of $\mathcal{D}$ is needed to separate points in $(0,1)$. It is clear that only $(1/4,1/2)$ separates $3/8$ and $1/2$ since neither of these points belongs to any other member of $\mathcal{D}$. Similarly, only $(1/2,3/4)$ separates $5/8$ and $1/2$. Now suppose $D \in \mathcal{D} \setminus \{(1/4,1/2),(1/2,3/4)\}$. Then there is an integer $n \geq 3$ such that D is either in $\mathcal{E}_n$ or replaces a member of $\mathcal{E}_n$. The following assertions are easily checked: If $D = (2^{-n}, 2\cdot 2^{-n})$, then only D separates $3\cdot 2^{-n-1}$ and $2\cdot 2^{-n}$. If $D = (1-2\cdot 2^{-n}, 1-2^{-n})$, then only D separates $1-3\cdot 2^{-n-1}$ and $1-2\cdot 2^{-n}$. If $D = (2^{-n}, 3\cdot 2^{-n})$, then only D separates $5\cdot 2^{-n-1}$ and $3\cdot 2^{-n}$. If $D = (1-3\cdot 2^{-n}, 1-2^{-n})$, then only D separates $1-5\cdot 2^{-n-1}$ and $1-3\cdot 2^{-n}$. Finally, if $D = ((k-1)\cdot 2^{-n}, k\cdot 2^{-n})$, where $4 \leq k \leq 2^n - 3$, then let x be either the left endpoint $(k-1)\cdot 2^{-n}$ if k is even or the right endpoint $k \cdot 2^{-n}$ if k is odd. Only D separates its midpoint $(2k-1)\cdot 2^{-n-1}$ and x. Hence, $\mathcal{D}$ is a family of open intervals with compact closure which is a minimal generating family for the Borel sets of $(0,1)$.

Remark. Let $\epsilon > 0$. Under the homeomorphism $f_\epsilon : (0,1) \to R$ given by

$$f_\epsilon(x) = \begin{cases} \epsilon \log_2(2x) & \text{if } 0 < x \le 1/2 \\ -\epsilon \log_2(2-2x) & \text{if } 1/2 < x < 1, \end{cases}$$

the minimal generating family of open sets $\mathcal{D}$ in Example 2 is mapped onto a minimal generating family of open sets $f(\mathcal{D})$ for which each member of $f(\mathcal{D})$ has length less than 2ϵ. Hence, a minimal generating family of open sets for $\mathcal{B}_R$ can have mesh as small as we please.

References

[1] M. Filipczak, *On generators for Borel sets*, Real Analysis Exchange, **13** (1987–88), 194–203.

[2] K. P. S. Bhaskara Rao and B. V. Rao, *Borel spaces*, Dissertationes Mathematicae, **190** (1981), 1-63.

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