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PATH DIFFERENTIATION IN BOREL THE SETTING

I. Introduction. A unifying approach to the study of a number of generalized derivatives was introduced by Bruckner, O'Malley and Thomson in [2]. Namely, a set $E(x) \subset R$ (the real line with the usual topology) is a path at $x \in R$ if x is a limit point of $E(x)$. A system of paths E, is a collection $\{ E(x) : x \in R \}$ (or it can be considered as a multifunction $E: x \to E(x)$), where each $E(x)$ is a path at x. If $f: R \to R$ is a function, then the extreme E-derivatives of f at a point x are defined as follows

$$\bar{f}'_E(x) = \limsup_{\substack{y \to x \\ y \in E(x)}} \frac{f(x)-f(y)}{x-y} \qquad \text{and} \qquad \underline{f}'_E(x) = \liminf_{\substack{y \to x \\ y \in E(x)}} \frac{f(x)-f(y)}{x-y} .$$

If $\bar{f}'_E(x) = \underline{f}'_E(x)$, their common value is called the E-derivative of f at x ($f'_E(x)$).

Many results concerning various properties of path derivatives are based on a system of paths satisfying some of the intersection conditions [2], [3] as well as on the descriptive theory of system of paths considered as multifunction [1], [5]. A. Alikhani-Koopaei in [1] uses the system of paths as a continuous compact - valued multifunction and he quotes the following example due to Laczkovich.

EXAMPLE 1. There is a Baire 2 function f and a perfect set P with $0 \in P$ so that the congruent extreme derivative of f with respect to P i.e., $\bar{f}'_E$ is not a Borel function where $E(x) = x+P$. Hence the multifunction E_f of all E-derived numbers of f is not Borel measurable (because of $\bar{f}'^{-1}_E((a,\infty)) = E^-_f((a,\infty)) := \{x: E_f(x) \cap (a,\infty) \neq \emptyset \}$). But there is a Baire 4 selection for E_f. Namely, if $\{y_n\}_{n=1}$ is a sequence in $P\setminus\{0\}$ so that $\lim y_n = 0$, then $g(x) = \lim_{n\to\infty} \sup \dfrac{f(g_n(x))-f(x)}{g_n(x)-x}$ is in Baire class 4, where $g_n(x) = y_n + x$ for all $x\in R$.

The purpose of this paper is to investigate the similar behavior of E_f for a lower Borel α system of paths E and a Baire β function f.

II. Basic definitions and notation

Throughout this paper A_α (M_α) denotes the family of all subsets of R of the Borel additive (multiplicative) class α. By a multifunction $F:R\to R^*$ we mean a function defined on R and whose values are subsets of R^* (the extended real line with the topology of the two-point compactification of R). We also admit empty values for F. Given multifunctions F and G, $F\subset G$ means that $F(x)\subset G(x)$ for all $x\in R$ and the multifunctions $F\cap G$ and $\bar{F}$ are defined by $F\cap G(x) = F(x)\cap G(x)$ and $\bar{F}(x) = \overline{F(x)}$ where $\overline{F(x)}$ denotes the closure of $F(x)$.

For $A\subset R^*$ we let

$$F^-(A) = \{x : F(x)\cap A \neq \emptyset\},$$
$$F^+(A) = \{x : F(x)\subset A\},$$
$$Gr(F) = \{(x,y) : y\in F(x)\} \text{ (graph of F).}$$

DEFINITION 2 (the semi Borel classification of multifunctions). A multifunction $F:R\to R^*$ is said to be lower Borel α ($F\in lB_\alpha$) (upper Borel α ($F\in uB_\alpha$)) if for any open set $V\subset R^*$ $F^-(V)\in A_\alpha$ ($F^+(V)\in A_\alpha$). F is said to be lower (upper) semicontinuous (briefly lsc (usc)) if $F\in lB_o$ ($F\in uB_o$). F is continuous if it is lsc and usc. Note that within single-valued multifunctions the semi Borel classification coincides with the Baire classification of functions.

If E is a system of paths and f is a function, then we denote by E_f the following non-empty and compact-valued multifunction from R to R^* defined as follows:

$E_f(x) = \{y \in R^*$: there is a sequence $\{x_n\}_{n=1}^{\infty}$ in $E(x) \setminus \{x\}$

so that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \dfrac{f(x_n)-f(x)}{x_n - x} = y \}$ i.e. E_f is the multifunction of all E-derived numbers of f.

III. Results.

LEMMA 3. Let $H: R \to R$ be an open graph multifunction. If $F: R \to R$ is lower Borel α, then $F \cap H$ is lower Borel α.

PROOF. Define $F_{Gr(H)}: R \to R \times R$ by $F_{Gr(F)}(x) = (\{x\} \times F(x)) \cap Gr(H)$.

Let $Gr(H) = \bigcup_{n=1}^{\infty} H_n \times K_n$, where H_n, K_n are open. If $G_1, G_2 \subset R$ are open, then $F^-_{Gr(H)}(G_1 \times G_2) = \bigcup_{n=1}^{\infty} ((F^-(K_n \cap G_2)) \cap H_n \cap G_1) \in A_\alpha$. The equality $(F \cap H)^-(G) = F^-_{Gr(H)}(R \times G)$ finishes the proof where $G \subset R$ is an arbitrary open set.

THEOREM 4. Let E be a lower Borel α $(\alpha > 0)$ system of paths with closed values (i.e. E is considered as a closed-valued multifunction in $\mathbb{B}_\alpha$). If f is a Baire β function $(\beta \geq 0)$, then there is an upper Borel $\alpha+\beta+1$ non-empty and compact-valued multifunction $S: R \to R^*$ such that $S \subset E_f$. Consequently if f has E-derivative (finite or infinite), then it is in Baire class $\alpha+\beta+1$.

PROOF. Define $H_n: R \to R$ by $H_n(x) = (x - \frac{1}{n} - \frac{\varepsilon}{n}, x - \frac{1}{n+1}) \cup$

$(x + \frac{1}{n+1}, x + \frac{1}{n} + \frac{\varepsilon}{n})$ $n = 1,2,3,\dots$ where ε is a positive constant. By Lemma 3, $E\cap H_n \in \mathbb{B}_\alpha$. Since $(\overline{E\cap H_n})^-(G) =$ $(E\cap H_n)^-(G)$ for any G open, $\overline{E\cap H_n} \in \mathbb{B}_\alpha$. Let $X_n = \{x : \overline{E\cap H_n}(x) \neq \emptyset\} =$ $\{x : (\overline{E\cap H_n})^-(R)\} \in A_\alpha$ and $A_\alpha(X_n) = \{A \subset X_n$: A is a set of the Borel additive class α with respect to $X_n\}$. Since $A_\alpha(X_n) =$ $\{ A\subset X_n : A = X_n \cap B$ for some $B \in A_\alpha \}$ and $X_\alpha \in A_\alpha$, there is a selection $g_n : X_n \to R$ for the restriction $\overline{E\cap H_n}$ to X_n so that $g_n^{-1}(G) \in A_\alpha$ for every open $G \subset R$ (see [4]).

Define $f_n : X_n \to R$ by $f_n(x) = \dfrac{f(g_n(x)) - f(x)}{g_n(x) - x}$. It is clear that $f_n^{-1}(G) \in A_{\alpha+\beta}$ for every open G. Since $\bigcup_{n=1}^{\infty} H_n =$ $(x-1-\varepsilon, x+1+\varepsilon)\setminus\{x\}$ and x is a limit point of $E(x)$, x is also a limit point of $\bigcup_{n=1}^{\infty} X_n$ for all $x \in R$. Consequently the following multifunction $S:R\to R^*$ defined by $S(x) = \{y \in R^*$: there is a subsequence $\{n_k\}_{k=1}^{\infty}$ with $g_{n_k}(x) \in X_{n_k}$ and $y = \lim_{k\to\infty} f_{n_k}(x)\}$ is non-empty and compact-valued. The equality $S^-(K) =$ $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} f_n^{-1}(V_k) \in M_{\alpha+\beta+1}$ (where K is an arbitrary closed set in R^* and V_k are open in R^* with $\bar{V}_{k+1} \subset V_k$, $\bigcap_{k=1}^{\infty} V_k = K$) finishes the proof since $S^+(R\setminus K) = R\setminus S^-(K)$.

REMARK 5. If f is continuous, then the assumption of Theorem 4 concerning closed values of E can be omitted because $E_f = \bar{E}_f$.

COROLLARY 6. Under the same conditions on E and f as in Theorem 4, there is a Baire $\alpha+\beta+2$ selection for E_f.

PROOF. Multifunction S from Theorem 4 is in $lB_{\alpha+\beta+2}$ and by [4] there is a Baire $\alpha+\beta+2$ selection for S.

Note that the selection theorem of Kuratowski and Ryll-Nardzewski [4] being used in the proof of Theorem 4 holds for $\alpha>0$. We state as corollary the following weaker version of it for $\alpha=0$.

COROLLARY 9. (An improvement of [1,Theorem 18]. If E is a closed-valued lower (upper) semicontinuous system of paths and f is in Baire class β ($\beta\geq 0$), then there is a non-empty compact-valued upper Borel $\beta+2$ multifunction $S\subset E_f$. If f has E-derivative (finite or infinite), then f'_E is a Baire $\beta+2$ function.

The assertion follows directly from Theorem 4 because of $E\in lB_1$.

As we saw in Example 1 E_f need not be Borel measurable. We now briefly discuss some special cases for which E_f behaves very nicely. The following theorems give the conditions on E and f for which E_f is in uB_1 or uB_2.

The proof of the next assertion is trivial and hence omitted.

LEMMA 10. Let E be an arbitrary system of paths and f let be a function. If K is a closed set in R^*, then

$$E_f^-(K) = \bigcap_{n=1}^{\infty} pr(f_o^{-1}(V_n)\cap Gr(E_n)) \quad \text{where}$$

$f_o((x,y)) = \frac{f(x)-f(y)}{x-y}$ for $x \neq y$,

$E_n(x) = E(x)\cap(x - \frac{1}{n}, x + \frac{1}{n})$ $x\in R$, $n = 1,2,3...$,

V_n are open in R^* such that $\bigcap_{n=1}^{\infty} V_n = K$, $\bar{V}_{n+1}\subset V_n$, $n=1,2,...$,

and $pr(A) = \{x$: there is y such that $(x,y)\in A\}$ where $A\subset R\times R$.

THEOREM 11. Let E be a lower semicontinuous system of paths and let f be a continuous function. Then E_f is an upper Borel one multifunction.

PROOF. We shall show that $A_n = pr(f_o^{-1}(V_n)\cap Gr(E_n))$ is open. Let $x_o\in A_n$. Then there is $y\in R$ such that $(x_o,y)\in f_o^{-1}(V_n)$ and $y\in E_n(x_o)$. Since f_o is continuous, there is $I\times J\ni(x_o,y)$ where I,J are open intervals such that $I\times J \subset f_o^{-1}(V_n)$. Since E_n is lsc, there is an open set $G\subset I$ with $x_o\in G$ such that $E_n(x)\cap J \neq \emptyset$ for any $x\in G$. Thus for any $x\in G$ there is $y_x \in E_n(x)\cap J$. Since $(x,y_x) \in Gr(E_n)\cap f_o^{-1}(V_n)$, we know that $x\in A_n$ for any $x\in G$. By Lemma 10, $E_f^-(K)\in M_1$. Hence $E_f\in uB_1$.

The following theorem follows directly from Lemma 10.

Theorem 12.

(a) If $Gr(E)$ is open and f is continuous, then $E_f \in uB_1$.

(b) If $Gr(E)$ is a F_σ set and f is Baire 1, then $E_f \in uB_2$.

COROLLARY 13

(a) If E is usc and f is continuous, then $E_f \in uB_2$.

(b) If E is usc with closed values and f is Baire 1, then $E_f \in uB_2$.

PROOF. (a) Since $\bar{E}$ is usc, $Gr(\bar{E})$ is closed. By Theorem 12(b), $E_f \in uB_2$ because $\bar{E}_f = E_f$.

(b) It follows from Theorem 12(b) since Gr(E) is closed.

The remaining cases are formulated as the following open problems.

PROBLEM 14. What is the semi Borel classification of E_f if

(a) $E \in lB_\alpha$ $(\alpha \le 1)$ and f is Baire 1,

(b) $E \in uB_1$ and f is Baire β $(\beta \le 1)$,

(c) $E \in lB_\alpha$ $(\alpha \ge 1)$ and f is continuous?

PROBLEM 15. Is there a Baire $\alpha+\beta+1$ selection for E_f $(\alpha,\beta \ge 0)$.

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