Real Analysis Exchange Vol 16 (1990-91)

K. P. S. Bhaskara Rao, Indian Statistical Institute, Bangalore, India

R. M. Shortt, Wesleyan University, Middletown, Connecticut 06457

Separation of points by families of intervals

Let X be a separable metric space. It has been shown (see [1] or [3]) that the Borel structure on X has a minimal generator, i.e. there is a family $\mathcal{F}$ of subsets of X such that the σ–field $\sigma(\mathcal{F})$ generated by $\mathcal{F}$ equals the Borel σ–field $\mathcal{B}(X)$, and such that $\sigma(\mathcal{F}_0) \neq \mathcal{B}(X)$ for any proper sub–collection $\mathcal{F}_0 \subsetneq \mathcal{F}$. Such minimal generators are necessarily countable, as follows from the well–known and easily proved

Lemma 1: Let $(X, \mathcal{B})$ be a measurable space with $\mathcal{B}$ countably generated. If $\mathcal{F} \subsetneq \mathcal{B}$ is such that $\sigma(\mathcal{F}) = \mathcal{B}$, then there is some countable $\mathcal{F}_0 \subsetneq \mathcal{F}$ such that $\sigma(\mathcal{F}_0) = \mathcal{B}$.

In particular, the real line $\mathbb{R}$ has a minimal Borel generator. In [1; p. 19], an argument was made attempting to show that no minimal generator for $\mathbb{R}$ could be constructed using solely intervals. The underlying premise was that a family of intervals is a generator if and only if the set of corresponding interval end–points were dense in $\mathbb{R}$. As pointed out by M. Filipczak [2], this premise is incorrect. Moreover, as we demonstrate, there is indeed a minimal generator for $\mathbb{R}$ comprising only intervals.

Let $\mathcal{F}$ be a family of subsets of a set X. Points $x, y \in X$ are separated by $\mathcal{F}$ if there is some $F \in \mathcal{F}$ such that either

$$x \in F \text{ and } y \notin F \qquad \text{or} \qquad y \in F \text{ and } x \notin F.$$

Say that $\mathcal{F}$ is a minimal separator if $\mathcal{F}$ separates each pair of distinct points from X, but no proper sub–family $\mathcal{F}_0 \subsetneq \mathcal{F}$ does.

The well–known 'Blackwell property' of $\mathbb{R}$ (see [1, p. 21]) can be stated as

Lemma 2: Let $\mathscr{F}$ be a countable sub–family of $\mathscr{B}(\mathbb{R})$. If $\mathscr{F}$ separates all pairs of points in $\mathbb{R}$, then $\sigma(\mathscr{F}) = \mathscr{B}(\mathbb{R})$.

Note: Lemma 2 answers Filipczak's question 2 [2, p. 202] in the negative. The same result holds for any complete separable metric space.

In Lemma 2, the countability of $\mathscr{F}$ is essential, because an uncountable sub–family of $\mathscr{B}(\mathbb{R})$ which separates all pairs of points in $\mathbb{R}$ need not be a generator of $\mathscr{B}(\mathbb{R})$. However, if $\mathscr{F}$ is an uncountable family of open intervals separating all pairs of points, Filipczak in [2] has shown that $\sigma(\mathscr{F}) = \mathscr{B}(\mathbb{R})$. As a consequence of our Theorem 1, Filipczak's result follows easily. Theorem 1 itself has a simple proof. Our Theorem 4 gives a result for closed non–degenerate intervals analogous to that of Theorem 1, which in turn gives Filipczak's result for closed intervals. Our example 1 shows that the analogous result for other types of intervals is not true. Let us raise the question of what can be said for families of open intervals which do not separate every pair of points: is there always a countable sub–family separating the same pairs? The answer is positive and requires an easy preliminary:

Lemma 3: Let $\mathscr{F}$ be a family of intervals. There is a countable sub–family $\mathscr{F}_0 \subseteq \mathscr{F}$ such that $\cap \mathscr{F}_0 = \cap \mathscr{F}$.

Theorem 1: Let $\mathscr{F}$ be a family of open intervals. Then there is a countable sub–family $\mathscr{F}_0 \subseteq \mathscr{F}$ separating the same pairs of points in $\mathscr{F}$.

Proof: For each pair of rational numbers $r < s$, let $\mathscr{F}_{rs}$ be a countable sub–family of $\mathscr{F}$ such that

1) $r, s \in I$ for each $I \in \mathscr{F}_{rs}$;

2) $\cap\ \mathcal{I}_{rs} = \cap\ \{I \in \mathcal{I}\colon r, s \in I\}$.

Lemma 3 implies that such a sub–family exists. Let $\mathcal{I}_0$ be the union of the collections $\mathcal{I}_{rs}$ as r and s range over pairs of rationals $r < s$.

We shall show that if $x < y$ are separated by $\mathcal{I}$, they are also separated by $\mathcal{I}_0$.

Case 1: There is some $J \in \mathcal{I}$ such that $x \in J$ and $y \notin J$. Choose rational numbers r and s in J such that $r < x < s$. Then

$\cap\ \{I \in \mathcal{I}\colon r, s \in I\} \subseteq J$,

so that $\cap\ \mathcal{I}_{rs} \subseteq J$. Since $x \in \cap\ \mathcal{I}_{rs}$ nd $y \notin \cap\ \mathcal{I}_{rs}$, it follows that x and y are separated by one of the intervals $\mathcal{I}_{rs}$.

Case 2: There is some $J \in \mathcal{I}$ such that $x \notin J$ and $y \in J$. The proof runs parallel to case 1, with $x < r < y < s$. Again, $\mathcal{I}_{rs}$ separates x and y.

Q.E.D.

Without substantial change, the proof of Theorem 1 can be used to establish the following somewhat more general result:

Theorem 2: Let $\mathcal{I}$ be a family of subsets of a set X. Suppose that

i) for each $\mathcal{I}_0 \subseteq \mathcal{I}$, there is a countable $\mathcal{I}_{00} \subseteq \mathcal{I}_0$ such that $\cap\ \mathcal{I}_{00} = \cap\ \mathcal{I}_0$;

ii) there is a countable family $\mathcal{G}$ of subsets of X such that whenever $x \in F$ for $F \in \mathcal{I}$, there is some $G \in \mathcal{G}$ with $x \in G \subseteq F$.

Then there is a countable sub–family of $\mathcal{I}$ separating the same pairs of points as $\mathcal{I}$.

<u>Corollary</u>: Let $\mathscr{F}$ be a collection of subsets of $\mathbb{R}^n$ of the form $I_1 \times \ldots \times I_n$, where each I_k is an open interval. Then there is a countable sub–family of $\mathscr{F}$ separating the same points as $\mathscr{F}$.

<u>Proof</u>: Apply Theorem 2, taking $\mathscr{G}$ as the collection of sets of the form $J_1 \times \ldots \times J_n$, with each J_k a closed interval with rational end–points.

Q.E.D.

Taking complements in Theorem 2 yields a dual result.

<u>Theorem 3</u>: Let $\mathscr{F}$ be a family of subsets of a set X. Suppose that

i) for each $\mathscr{F}_0 \subseteq \mathscr{F}$, there is some countable $\mathscr{F}_{00} \subseteq \mathscr{F}_0$ such that $\cup\, \mathscr{F}_{00} = \cup\, \mathscr{F}_0$;

ii) there is a countable family $\mathscr{G}$ of subsets of X such that whenever $y \notin F$ for $F \in \mathscr{F}$, there is some $G \in \mathscr{G}$ with $y \in G$ and $F \cap G = \phi$.

Then there is a countable sub–family of $\mathscr{F}$ separating the same points as $\mathscr{F}$.

Theorem 3 may be applied to families of closed intervals.

<u>Lemma 4</u>: Let $\mathscr{F}$ be a family of non–degenerate intervals (of any type). There is a countable sub–family $\mathscr{F}_0 \subseteq \mathscr{F}$ such that $\cup\, \mathscr{F}_0 = \cup \mathscr{F}$.

<u>Indication</u>: For $x, y \in \cup\, \mathscr{F}$, write $x \sim y$, if there are $I_1 I_2 \ldots I_k$ in $\mathscr{F}$ with $x \in I_1$, $y \in I_k$, and $I_j \cap I_{j+1} \neq \phi$ for $1 \leq j \leq k-1$. The equivalence classes of $\sim$ are intervals,

each is a countable union of intervals from $\mathcal{I}$, and there are at most countably many such classes. The conclusion follows.

Theorem 4: Let $\mathcal{I}$ be a family of non–degenerate closed intervals. Then there is a countable family $\mathcal{I}_0 \subseteq \mathcal{I}$ separating the same pairs of points as $\mathcal{I}$.

Note: The condition of non–degeneracy cannot be eliminated, as the family $\mathcal{I} = \{\{x\}: x \in \mathbb{R}\}$ illustrates.

Indication: This follows from Theorem 3 and Lemma 4.

However, in contrast to the corollary to Theorem 2, we offer the following example. Note that property i) fails for closed boxes in $\mathbb{R}^n$.

Example 1: Let $\mathcal{I}$ be the collection of all subsets of $\mathbb{R}^2$ of the form $[x , y] \times [-x , z]$, with $x < y$ and $-x < z$. Then $\mathcal{I}$ separates points on the anti–diagonal $Y = -X$, but the same cannot be said for any countable $\mathcal{I}_0 \subseteq \mathcal{I}$.

Also, the same type of result does not obtain for half–open intervals, as we show.

Example 2: Consider the family $\mathcal{I} = \{(-x, x] : x \in \mathbb{R}\}$ of half–open intervals. Then $\mathcal{I}$ separates any pair of points in $\mathbb{R}$, but for each $x \neq 0$, the interval $(-x,x]$ is the only member of the family $\mathcal{I}$ separating x and $-x$. Thus $\mathcal{I}$ is a minimal separator and contains no countable $\mathcal{I}_0$ separating the same pairs of points.

It is interesting to observe that $\mathcal{I}$ cannot be a generator for the Borel σ–field on $\mathbb{R}$. In fact $\sigma(\mathcal{I})$ comprises all sets $B \triangle C$, where B is a symmetric Borel set (i.e. $B = -B$), and C is countable.

As a consequence of Theorems 1 and 4, we have the following remarkable result.

Theorem 5: If $\mathcal{I}$ is any family of either open or closed, non–degenerate intervals, then there is a countable sub–family $\mathcal{I}_0 \subsetneq \mathcal{I}$ such that $\sigma(\mathcal{I}_0) = \sigma(\mathcal{I})$. In particular, $\sigma(\mathcal{I})$ is countably generated.

Proof: Using Theorems 1 and 2, obtain a countable $\mathcal{I}_0 \subsetneq \mathcal{I}$ such that $\mathcal{I}_0$ and $\mathcal{I}$ separate the same pairs of points. This implies that any $I \in \mathcal{I}$ is a union of atoms of $\sigma(\mathcal{I}_0)$. By the Blackwell property for $\sigma(\mathcal{I}_0)$ (see [1]), $I \in \sigma(\mathcal{I}_0)$.

Q.E.D.

We now establish the existence of a minimal generator for $\mathbb{R}$ comprising only intervals.

Theorem 6: There is a minimal generator $\mathcal{I}$ whose elements are half open intervals.

Proof: The members of $\mathcal{I}$ will be the intervals

$$I(m,n) = (m2^{-n}, m2^{-n} + 2^{-n-1}],$$

where m and n are integers.

Claim 1: Of the intervals in $\mathcal{I}$, $I(m,n)$ and only $I(m,n)$ separates the pair of points $(2m + 1)2^{-n-1}$ and $(2m + 2)2^{-n-1}$.

Proof of claim: Suppose that $I(M,N)$ separates these points. Then either

$$(2m + 1)2^{-n-1} \leq M2^{-N} < (2m + 2)2^{-n-1} \leq M2^{-N} + 2^{-N-1}$$

or

$$M2^{-N} < (2m+1)2^{-n-1} \le M2^{-N} + 2^{-N-1} < (2m+2)2^{-n-1}.$$

We show that the first case cannot happen. For, if $n \le N$, then

$$2M < (2m+2)2^{N-n} \le 2M+1,$$

where the middle term is an even integer, a contradiction. On the other hand, if $n > N$, then

$$2m+1 \le M2^{n+1-N} < 2m+2,$$

where once again the middle term is an even integer, a contradiction.

We thus suppose that the second case happens. Now, if $n < N$, then

$$2M < (2m+1)2^{N-n} \le 2M+1,$$

where the middle term is an even integer, a contradiction. But if $n > N$, then

$$2m+1 \le M2^{n+1-N} + 2^{n-N} < 2m+2,$$

where the middle term is an even integer, a contradiction. Hence, $n = N$, and we have

$$2M < 2m+1 \le 2M+1 < 2m+2,$$

so that $m = M$.

Claim 2: Let x and y be distinct real numbers. Then x and y are separated by the family of all intervals I(m,n).

Proof of claim: We may assume $y < x$. Let N be the smallest integer such that $2^{-N-1} < x - y$, and let M_y be the largest integer such that $M_y 2^{-N+1} < y$. Then $y \leq (M_y + 1)2^{-N+1}$. There are four cases to consider for y.

Case 1: $y \in I(2M_y,N) = (2M_y 2^{-N}, 2M_y 2^{-N} + 2^{-N-1}] = (M_y 2^{-N+1}, M_y 2^{-N+1} + 2^{-N-1}]$. Then $I(2M_y,N)$ separates y and x.

Case 2: $y \in (M_y 2^{-N+1} + 2^{-N-1}, M_y 2^{-N+1} + 2^{-N}]$. Then x lies to the right of this interval, so that $y \in I(M_y,N-1)$ and $x \notin I(M_y,N-1)$.

Case 3: $y \in I(2M_y + 1, N)$. Then this interval separates y and z.

Case 4: $y \in (2M_y + 1)2^{-N} + 2^{-N-1},(M_y + 1)2^{-N+1}]$. Since $2^{-N-1} < x - y \leq 2^{-N}$, we have $x \in I(M_y + 1, N - 1)$ so that $I(M_y + 1, N - 1)$ separates x and y.

Thus $\mathscr{I}$ is a minimal separator and hence (lemma 2) a minimal generator for $\mathscr{B}(\mathbb{R})$.

Q.E.D.

Note: The referee of this paper intercepted critical errors in an earlier version of Theorem 6, and the proof given above makes use of his suggestions. Also, S. Solecki of Wrocław has informed the authors that the system $\{ (n2^k, (n+1)\, 2^k):\ n, k \in \mathbb{Z} \}$ is a minimal separator for $\mathscr{B}(\mathbb{R})$ comprising only open intervals.

Filipczak [2] states a number of results for spaces other than $\mathbb{R}$, some for general

Hausdorff spaces. Indeed, it is natural to ask whether Theorems 1 and 2 hold for open, convex subsets of $\mathbb{R}^n$. The answer is no:

Example 3: Let C be a circle in the plane $\mathbb{R}^2$. Let $\mathcal{F}_1$ be the collection of all open half–planes whose boundary is tangent to C and which contain the interior C. Let $\mathcal{F}_2$ be the collection of all open, convex sets interior to C. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ separates points of $\mathbb{R}^2$, but for each countable $\mathcal{F}_0 \subseteq \mathcal{F}_1$ there are points x,y on the circumference of C not separated by $\mathcal{F}_0$.

The example answers Filipczak's Question 1 in [2; p. 202], with reference to her Theorem 3. As for the same question with reference to her Theorems 4 and 5, we offer the following

Example 4: Consider (0,1) with the discrete topology and let $X = (0,1) \cup \{\infty\}$ be its one–point compactification. For each rational $r \in (0,1)$ define $H_r = (0,r) \cup \{\infty\}$. The family $\mathcal{F} = \{H_r : r \text{ rational}\}$ is a countable family of compact subsets of X separating all pairs of points in X. But $\sigma(\mathcal{F}) \neq \mathcal{B}(X)$. Thus, condition (**) cannot replace (***) in Theorem 4 of [2]. Taking complements of sets in $\mathcal{F}$ shows that (**) cannot replace (***) in Theorem 5 of [2].

Acknowledgement

Thanks are hereby expressed to A. Fieldsteel, A. Molitor and L. C. Robertson for helpful conversations, and to S. Solecki for informative remarks.

The authors also wish to thank the referees for their careful reading.

[1] K. P. S. Bhaskara Rao and B. V. Rao, Borel spaces, Dissertationes Math. CXC (1981) 1–63

[2] M. Filipczak, On generators for Borel sets, Real Analysis Exchange 13 (1987–88) 194–203

[3] B. V. Rao, On Borel structures, Colloq. Math. 21 (1970) 199–204

Received December 26, 1989