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ALGEBRAIC STRUCTURES GENERATED BY $\mathcal{T}_d$-QUASI CONTINUOUS AND ALMOST CONTINUOUS FUNCTIONS ON R^m.

I. PRELIMINAIRIES. Let $(X,\mathcal{T})$ be any topological space and $\mathcal{R}$ denotes the real line. A function $f : X\to\mathcal{R}$ is called $\mathcal{T}$-quasi continuous ($\mathcal{T}$-cliquish) at a point $x\in X$ iff for every $\varepsilon>0$ and for every neighbourhood $U\in\mathcal{T}$ of the point x there exists a $\mathcal{T}$-open set V such that $\emptyset\neq V\subset U$ and $|f(u) - f(x)|<\varepsilon$ for every $u\in V$ ($\operatorname{osc}_V f<\varepsilon$). A function f is $\mathcal{T}$-quasi continuous ($\mathcal{T}$-cliquish) on X iff f is $\mathcal{T}$-quasi continuous ($\mathcal{T}$-cliquish) at every point of X ([1]).

Let $X=R^m$. We shall use the following differentiation basis ([8]). For every $k=1,2,\ldots$ let $\mathcal{P}_k$ be the family of all m-dimensional intervals of the form

$$\left[\frac{i_1-1}{2^k}, \frac{i_1}{2^k}\right) \times \left[\frac{i_2-1}{2^k}, \frac{i_2}{2^k}\right) \times \ldots \times \left[\frac{i_m-1}{2^k}, \frac{i_m}{2^k}\right),$$

where $i_1,i_2,\ldots,i_m$ are integers. Let $\mathcal{P}=\bigcup_{k=1}^{\infty}\mathcal{P}_k$.

Let $A\subset R^m$ be a measurable (L) (i.m. measurable in the sense of Lebesgue) set. For $x\in R^m$ we can define the upper density of A at a point by

$$\bar{d}(A,x) = \varlimsup_{\substack{P\to x\\ P\in\mathcal{P}}} \frac{|A\cap P|}{|P|},$$

where $|A|$ denotes m-dimensional measure (L) of A and the symbol $P\to x$ denotes that $x\in P$ and the diameter of P tends to zero. If $\bar{d}(R^m-A, x) = 0$ then we say that x is a density point of A.

Let $\varphi(A)$ denote the set of all density points of A. Notice that the basis $\mathcal{P}$ has the density property , i.e. for every measurable set A and for almost every $x\in A$, x is a density point of A ([3]). The family of all measurable (L) sets A with $A\subset\varphi(A)$ forms a topology on R^m. This topology is called the density topology $\mathcal{T}_d$ ([10]). A function $f:R^m\to R$ is approximately continuous at a point x iff it is $\mathcal{T}_d$-continuous at this point.

In a paper [5] it is proven that for every $\mathcal{T}_d$-cliquish function $f:R^m\to R$ there are $\mathcal{T}_d$-quasi continuous functions

$$g,h,g_1,g_2,g_3,g_4,f_1,f_2,\dots,f_n,\dots : R^m\to R$$

such that $f=g+h$, $f=\min(\max(g_1,g_2),\max(g_3,g_4))$ and $f=\lim_{n\to\infty} f_n$.

In this article we prove that the functions $g,h,g_1,g_2,g_3,g_4,$ $f_1,f_2,\dots,f_n,\dots$ can be also almost continuous.

A function $f:R^m\to R$ is almost continuous iff for every open set $U\subset R^m\times R$ containing the graph $\Gamma(f)$ of the function f there exists a continuous function $g:R^m\to R$ such that $\Gamma(g)\subset U$ ([13]).

II. BASIC LEMMATA.

Lemma 1.([5]). A function $f:R^m\to R$ is $\mathcal{T}_d$-cliquish iff f is measurable (L).

Lemma 2.(compare [5]). A measurable (L) function $f:R^m\to R$ is $\mathcal{T}_d$-quasi continuous at a point x iff for every $\gamma>0$ we have $\bar{d}(f^{-1}(f(x)-\gamma,f(x)+\gamma)\ ,\ x)>0$.

Lemma 3.([5]). Assume that A is a G_δ set of measure (L) zero, G is an open set and $A\subset G$. Then there exists a sequence of pairwise disjoint measurable (L) sets $A_n\subset G-A$ ($n=0,1,2,\dots$) such that $\bigcup_{n=0}^{\infty} A_n = G-A$, $\bar{d}(A_n,x)>0$ for every $x\in A\cup A_n$ ($n=0,1,2,\dots$) and $\bar{d}((R^m-G)\cup A_0\ ,\ x)>0$ for each $x\in R^m-G$.

A set $K \subset R^m \times R$ is called a blocking set for a function $f: R^m \to R$ iff K is a closed set, $K \cap \Gamma(f) = \emptyset$ and for every continuous function $g: R^m \to R$ we have $K \cap \Gamma(g) \neq \emptyset$.
A set K is a minimal blocking set of f in $R^m \times R$ if K is a blocking set of f and no proper subset of $\mathcal{K}$ is a blocking set of f in $R^m \times R$ ([7]).

Lemma 4 .(compare [7]). A function $f: R^m \to R$ is not almost continuous iff there exists a minimal blocking set K of f in $R^m \times R$. If K is a minimal blocking set for any function $f: R^m \to R$ then $p_x(K)$ is nondegenerated continuum and $p_y(K) = R$, when $p_x(K)$ and $p_y(K)$ denote the projections of K on the space R^m and R respectively.

Lemma 5. Suppose that $A \subset R$ is a residual G_δ set of measure (L) zero ([11] p.15, th.16). Let $B = \bigcup_{i=1}^{m} (R^{i-1} \times A) \times R^{m-i}$ and let C denotes any nondegenerated continuum contained in R^m. Then the set $C \cap B$ is not countable.

Proof. From the paper [14] results that there exists a sequence of pairwise disjoint closed sets A_n ($n=1,2,\ldots$) such that $R - A = \bigcup_{n=1}^{\infty} A_n$. Then

$$R^m - B = \underbrace{(R-A) \times (R-A) \times \ldots \times (R-A)}_{m \text{ times}} =$$

$$= \left(\bigcup_{i_1=1}^{\infty} A_{i_1} \right) \times \left(\bigcup_{i_2=1}^{\infty} A_{i_2} \right) \times \ldots \times \left(\bigcup_{i_m=1}^{\infty} A_{i_m} \right) =$$

$$= \bigcup_{i_1, i_2, \ldots, i_m = 1}^{\infty} \left(A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m} \right) .$$

Hence

(1) $$C = (C \cap B) \cup (C - B) = (C \cap B) \cup [\, C \cap (R^m - B)\,] =$$

$$= (C \cap B) \cup \left[\, C \cap \bigcup_{i_1, i_2, \ldots, i_m = 1}^{\infty} (A_{i_1} \times A_{i_2} \times \ldots \times A_{i_m}) \right] =$$

$$= (C\cap B) \cup \bigcup_{i_1,i_2,\dots,i_m=1}^{\infty} [C \cap (A_{i_1} \times A_{i_2} \times \dots \times A_{i_m})].$$

If $C\cap B$ is a countable set then from the equality (1) follows that C is the countable union of pairwise disjoint closed and nonempty sets. But the continuum C can not be a countable union of pairwise disjoint closed nonempty sets (see e.g. [4], p. 240 th. 14). Hence the set $C\cap B$ is not countable and the proof of. the lemma is finished.

III. **THEOREMS.** We assume the Continuum Hipotesis. Arange all blocking sets in the space $R^m\times R$ in a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \omega_1$ (where ω_1 denotes the first nondenumerable ordinal number).
Let $\{w_n\}_{n=0}^{\infty}$ be an ennumeration of all rationals such that $w_i \neq w_j$ if $i\neq j$ ($i,j=0,1,\dots$).

Theorem 1. Every $\mathcal{T}_d$-cliquish function $f : R^m\to R$ is the sum of two $\mathcal{T}_d$-quasi continuous and almost continuous functions $g,h : R^m\to R$.

Proof. From the lemma 1 we have that f is a measurable (L) function. Since the collection $\mathcal{P}$ has the density property , f is an almost everywhere $\mathcal{T}_d$-continuous function. Let D be the set of all points at which f is not $\mathcal{T}_d$-continuous and let B be a set from the lemma 5. There is a G_δ set E of measure (L) zero such that $D \cup B \subset E$. Let A_n ($n=0,1,\dots$) be the sets from the lemma 3, where $A=E$ and $G=R^m$.

For every $\alpha < \omega_1$ we choose two points

$$(a_\alpha^1 , b_\alpha^1) , (a_\alpha^2 , b_\alpha^2) \in K_\alpha$$

such that

$$- \; a_\alpha^1 , a_\alpha^2 \in B \quad (\alpha < \omega_1)$$

$$- \; a_\beta^i \neq a_\gamma^j \text{ if } \beta \neq \gamma \text{ or } i \neq j \quad (\beta,\gamma < \omega_1,\ i,j=1,2)$$

The choise of points satisfying these conditions is possible because from the lemma 5 the set $B \cap p_x(K_\alpha)$ is not countable. Let

$$\Omega^1 = \{ a^1_\alpha : \alpha < \omega_1 \} \quad \text{and} \quad \Omega^2 = \{ a^2_\alpha , \alpha < \omega_1 \}$$

Define

$$g(x) = \begin{cases} f(x) & \text{for } x \in E - \bigcup_{i=1}^{2} \Omega^i \\ b^1_\alpha & \text{for } x = a^1_\alpha \quad (\alpha < \omega_1) \\ f(x) - b^2_\alpha & \text{for } x = a^2_\alpha \quad (\alpha < \omega_1) \\ w_n & \text{for } x \in A_{2n} \quad (n=0,1,2,\ldots) \\ f(x) - w_n & \text{for } x \in A_{2n-1} \quad (n=1,2,\ldots) \end{cases}$$

and

$$h(x) = \begin{cases} 0 & \text{for } x \in E - \bigcup_{i=1}^{2} \Omega^i \\ f(x) - b^1_\alpha & \text{for } x = a^1_\alpha \quad (\alpha < \omega_1) \\ b^2_\alpha & \text{for } x = a^2_\alpha \quad (\alpha < \omega_1) \\ f(x) - w_n & \text{for } x \in A_{2n} \quad (n=0,1,2,\ldots) \\ w_n & \text{for } x \in A_{2n-1} \quad (n=1,2,\ldots) \end{cases}$$

Obviously functions g, h are $\mathcal{T}_d$-quasi continuous and $f=g+h$. Because the graphs $\Gamma(g)$ and $\Gamma(h)$ cut all the blocking sets K_α ($\alpha < \omega_1$), so from the lemma 4 the functions g, h are almost continuous.

Theorem 2. Every $\mathcal{T}_d$-cliquish function $f: R^m \to R$ is the limit of sequence of $\mathcal{T}_d$-quasi continuous and almost continuous functins $f_n : R^m \to R$ $(n=1,2,\ldots)$.

Proof. Let D be the set of all points at which the function f is not $\mathcal{T}_d$-continuous and let B be the set from the lemma 5. There exists a G_δ set E of measure (L) zero such that $D \cup B \subset E$. Let sets A_n $(n=0,1,2,\ldots)$ satisfy the conditions of the lemma 3, where $G = R^m$ and $A=E$.
For every $\alpha < \omega_1$ there exist points

$$(a^i_\alpha , b^i_\alpha) \in K_\alpha \quad (i=1,2,\ldots)$$

such that

$$- \; a^{i}_{\alpha} \in B \quad (\alpha < \omega_1 \; ; \; i=1,2,\dots) \; ;$$

$$- \; a^{i}_{\beta} \neq a^{j}_{\gamma} \quad \text{if} \quad \beta \neq \gamma \quad \text{or} \quad i \neq j \; (\beta,\gamma < \omega_1 \text{ and } i,j=1,2,\dots).$$

There is a sequence of open sets G_n such that

$$E = \bigcap_{n=1}^{\infty} G_n \quad \text{and} \quad G_1 \supset G_2 \supset \dots \supset G_n \supset \dots \; .$$

For every $n=1,2,\dots$ there is a sequence $\{A^n_k\}_{k=0}^{\infty}$ of measurable (L) sets which are the same as in the lemma 3, where $A_k = A^n_k$, $G = G_n$ and $E = A$.

Let, for $n=1,2,\dots$,

$$f_n(x) = \begin{cases} b^k_{\alpha} & \text{for } x = a^k_{\alpha} \; ; \; k > n \text{ and } \alpha < \omega_1 \\ w_k & \text{for } x \in A^n_k \; (k=1,2,\dots) \\ f(x) & \text{in the remaining case} \; . \end{cases}$$

Every function f_n $(n=1,2,\dots)$ is $\mathcal{T}_d$-quasi continuous and almost continuous and $f = \lim_{n\to\infty} f_n$, so the proof is finished.

Theorem 3. If $f: R^m \to R$ is $\mathcal{T}_d$-cliquish function, then there are four $\mathcal{T}_d$-quasi continuous and almost continous functions $f_1, f_2, f_3, f_4 : R^m \to R$ such that

$$f = \min \{ \max(f_1, f_2) \; , \; \max(f_3, f_4) \}.$$

Proof. Let D be the set of all points at which the function f is not $\mathcal{T}_d$-continuous and let B be the set from the lemma 5. There exists a G_{δ} set E of measure (L) zero such that $D \cup B \subset E$. Let the sets A_n $(n=0,1,2,\dots)$ satisfy the conditions of the lemma 3 (for $G=R^m$ and $A=E$). For every $\alpha < \omega_1$ choose four points

$$\{ a^i_{\alpha} , b^i_{\alpha} \} \in K_{\alpha} \quad (i=1,2,3,4)$$

such that

$$- \; a^i_{\alpha} \in B \quad (i=1,2,3,4) \; ;$$

$$- \; a^i_{\beta} \neq a^j_{\gamma} \quad \text{for } \beta \neq \gamma \; (\beta,\gamma < \omega_1) \quad \text{or } i \neq j \; (i,j=1,2,3,4)$$

For $i=1,2,3,4$ define the functions

$$f_i(x) = \begin{cases} \omega_n & \text{for } x \in \bigcup_{n=0}^{\infty} A_{4n+i} \\ b_\alpha^i & \text{for } x = a_\alpha^i \quad (\alpha < \omega_1) \\ f(x) & \text{in another case} \end{cases}$$

All functions f_i $(i=1,2,3,4)$ are $\mathcal{T}_d$-quasi continuous and almost continuous.

If $g_1 = \max (f_1, f_2)$ and $g_2 = \max (f_3, f_4)$ then, for $\Omega^i = \{ a_\alpha^i , \alpha < \omega_1 \}$ $(i=1,2,3,4)$, we have

$$g_1(x) = f(x) \quad \text{for} \quad x \in \bigcup_{n=0}^{\infty} \bigcup_{i=3}^{4} A_{4n+i} \cup \bigcup_{i=3}^{4} \Omega^i ,$$

$$g_1(x) \geq f(x) \quad \text{for every} \quad x \in R^m ,$$

$$g_2(x) = f(x) \quad \text{for} \quad x \in \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2} A_{4n+i} \cup \bigcup_{i=1}^{2} \Omega^i \quad \text{and}$$

$$g_2(x) \geq f(x) \quad \text{for every} \quad x \in R^m .$$

Consequently $f = \min (g_1, g_2)$ and the proof is finished.

IV. **REMARK.** Let $f: R^m \to R$ be an almost continuous function. It is known ([13], Proposition 2 and Proposition 3) that for every closed connected nonempty set $A \subset R^m$ the function $f/_A$ is almost continuous and $\Gamma(f/_A)$ is connected set. So every almost continuous function $f: R^m \to R$ has the Darboux property in the Pawlak's sense ([12]) and in the Mišik's sense ([9]). In the case $m=1$ the function f has the Darboux property. So from the theorems 1,2,3 result the theorems 1,2,3 of the paper [6] respectively.

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