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A Theory of Integration for Cardinal Algebras

CONTENTS

§0. Introduction

The origin of the theory of cardinal algebras lies in work of Tarski, whose wonderful treatise [15] is the subject's central reference. The original premise seems to have been that addition of cardinal numbers might profitably be studied from an axiomatic point of view making no *a priori* use of the notion of "set". (This fits well with Tarski's suspicion that the Zermelo-Fraenkel set theory was too limited in scope.) At the same time, Tarski realised that a number of applications of the idea were possible in measure theory, descriptive set theory, and for the study of various ordered algebraic structures (lattices, Boolean algebras and lattice-ordered semi-groups). Some of these applications may be found in [15], but we mention [2], [4], [5], [6], [9], [10], [11] as a sampler.

Despite the fact that Tarski did apply cardinal algebras to the theory of countably additive measures and even alluded to the possibility of defining finitely additive measures on a cardinal algebra [15: section 14], he never pursued the idea of countably additive measures on a cardinal algebra, i.e. cardinal algebra homomorphisms from A to $[0, \infty]$. The collection A^* of all such homomorphisms forms a cardinal algebra "dual" to A. This train of thought is followed in [12], where a rudimentary duality theory for cardinal algebras is worked out.

As with most duality theories, there is a natural evaluation map $T : A \to A^{**}$ defined by $T(a)(\mu) = \mu(a)$. Although T need not map onto A, it can be shown (lemma 5.1) that under some hypotheses, a functional $U \in A^{**}$ is absolutely continuous (in fact, equivalent, in the measuretheoretic sense) with respect to $T(a)$ for some $a \in A$. If some sort of Radon-Nikodym Theorem were available, we could then perhaps write

$$U(\mu) = \int_{\mu} f \, dT(a).$$

To interpret such an expression, we need to develop an integration theory for measures on a cardinal algebra.

The main question to be answered in such a development concerns the proper domain for the functions to be integrated. In [5], Fillmore used the Stone space of a certain Boolean sub-algebra of A as a domain on which to represent elements of A as functions (e.g. his Theorem 3.11). This technique, which applies only under very special hypotheses, will not work here, but can be replaced by the following device: consider a Stone dual of the distributive lattice $\mathrm{Idl}(A)$, the algebra of all ideals of A. This is our approach, although we make use of Priestley's ordered-space version of the Stone dual space [7], since it results in the technically more pleasing environment of a compact Hausdorff space. Functions to be integrated are defined on this compact dual space; measures $\mu \in A^*$ and elements $c \in A$ together induce a Baire measure on the dual space, and

$$(*) \qquad \int_{c} f \, d\mu$$

is defined as an ordinary (Lebesgue) integral of f over this dual space. The expression (*) becomes a tri-linear form in the arguments c, f, μ.

It is the author's belief that cardinal algebras form a very natural, technically pleasing setting for measure theory: very little power is lost with the omission of Boolean complementation, and Tarski's refinement postulate is, for many arguments, exactly the required tool.

§1. Preliminary results

A cardinal algebra is a set A together with a commutative and associative operation of countable sum

$$(a_0, a_1, \ldots) \longrightarrow a_0 + a_1 + a_2 + \cdots$$

and a distinguished element $0 \in A$ which serves as an additive identity; furthermore, the system $(A, +, 0)$ is assumed to satisfy two key axioms:

Refinement postulate: If $a + b = \Sigma c_n$ in A, then one may write $a = \sum a_n$ and $b = \sum b_n$ in such a way that $a_n + b_n = c_n$ for each n.

Remainder postulate: If $a_n = b_n + a_{n+1}$ for $n = 0, 1, \ldots,$ then there is some $c \in A$ such that

$$a_n = c + b_n + b_{n+1} + \cdots$$

for each n.

We shall make use of a number of basic results concerning the arithmetic theory of cardinal algebras. All of these may be found in the first few chapters of Tarski's treatise [15], but a few have been selected for special mention.

1.1 General refinement lemma: Suppose that $\sum a_n = \sum b_m$ in a cardinal algebra A. Then there are elements d_{nm} in A such that

$$a_n = \sum_m d_{nm} \qquad \text{(each } n)$$

$$b_m = \sum_n d_{nm} \qquad \text{(each } m\text{)}.$$

Indication: This is Theorem 2.1 in [15].

In a cardinal algebra A, we use the following notational conventions:

$$0a = 0$$

$$na = a + a + \cdots + a \qquad (n \text{ times})$$

$$\omega a = a + a + a + \cdots .$$

Write $a \leq b$ in case there is some $c \in A$ with $b = a + c$. The relation $\leq$ is a partial order on A [15; 1.31].

1.2 Lemma: Suppose that $a \leq \sum b_n$ in a cardinal algebra A. Then there are elements a_n in A such that $a = \sum a_n$ and $a_n \leq b_n$.

Indication: This follows easily from the refinement postulate. It is Corollary 2.2 in [15].

1.3 Interpolation lemma: Suppose that a_n and b_n are sequences of elements in a cardinal algebra A such that $a_n \leq b_m$ for each n and m. Then there is some $c \in A$ such that $a_n \leq c \leq b_m$ for each n and m.

Indication: Theorem 2.28 in [15].

Cardinal algebras occur throughout mathematics: examples occur in algebra, analysis and the theory of partially ordered structures. Here are a few.

1.4 Example: $A = [0, \infty]$ under ordinary addition of extended real numbers.

1.5 Example: $A = \{0, 1, 2, \ldots, \infty\}$, again under ordinary addition.

1.6 Example: Any countably complete and countably distributive lattice A with least element 0 and $\sum a_n$ defined as the supremum of the elements a_n. See 15.10 in [15].

1.7 Example: The collection A of all Borel-isomorphism type of separable metric spaces: if $t_1\ t_2 \ldots$ are the types of spaces $X_1\ X_2 \ldots$, then $t_1 + t_2 + \cdots$ is the type of the topological sum of the X_n. [15; pp. 234-5].

1.8 Example: Let $(X, \mathcal{B})$ be a measurable space and let A be the collection of all $\mathcal{B}$-measurable functions $f : X \to [0, \infty]$. Under point-wise addition of functions, A becomes a cardinal algebra. This is a result of Chuaqui [1].

1.9 Example: Let $(X, \mathcal{B})$ be a measurable space and let A be the collection of all measures on $(X, \mathcal{B})$. It follows from results in [12] that A is a cardinal algebra.

1.10 Example: Let $A_1\ A_2 \ldots$ be cardinal algebras. Then so is $A = A_1 \times A_2 \times \cdots$ under co-ordinate-wise sum. See [15: Theorem 6.12].

If A is a cardinal algebra, then $F \subseteq A$ is a semi-ideal of A if

1) F is non-empty;

2) $a \in F$ and $b \leq a$ imply $b \in F$;

3) $a, b \in F$ implies $a + b \in F$.

If 3) can be replaced by

3') $a_1\ a_2 \ldots \in F$ implies $a_1 + a_2 + \cdots \in F$,

then F is an ideal of A. The collection $\mathrm{Idl}(A)$ of all ideals of A becomes itself a cardinal algebra: the addition operation is defined by

$$I_1 + I_2 + \cdots = \{a_1 + a_2 + \cdots : a_n \in I_n\}.$$

(See chapters 9 and 10 of [15] for details.) When partially ordered by set-theoretic inclusion, $\mathrm{Idl}(A)$ become a distributive lattice: infimum and supremum are given by

$$I_1 \wedge I_2 = I_1 \cap I_2 \qquad\qquad I_1 \vee I_2 = I_1 + I_2$$

whilst $\{0\}$ and A are, respectively, the least and greatest elements of $\mathrm{Idl}(A)$. The smallest ideal containing $a \in A$ is

$$I(a) = \{b \in A : b \leq \omega a\},$$

the principal ideal generated by a.

Given cardinal algebras A_1 and A_2, a function $\alpha : A_1 \to A_2$ is a (cardinal algebra) homomorphism if

$$\alpha(0) = 0 \quad \text{and} \quad \alpha(a_1 + a_2 + \cdots) = \alpha(a_1) + \alpha(a_2) + \cdots .$$

A homomorphism $\mu : A \to [0, \infty]$ is a <u>measure</u>. The collection of all measures on A we denote by A^*. Under point-wise addition of functions, A^* becomes a cardinal algebra [12: 2.1]. Given μ in A^* and an ideal $I \subseteq A$, we define

$$\mu_I(a) = \sup\{\mu(b) : b \leq a \text{ and } b \in I\}.$$

Then we have

<u>1.11 Lemma</u>: The function μ_I is a measure on A.

<u>Proof</u>: Certainly $\mu_I(0) = 0$. Suppose that $a = a_1 + a_2 + \cdots$ and that $b \leq a$ with $b \in I$. From lemma 1.2 it follows that $b = \sum b_n$ with $b_n \leq a_n$. Thus

$$\mu(b) = \mu(b_1) + \mu(b_2) + \cdots \leq \mu_I(a_1) + \mu_I(a_2) + \cdots .$$

Taking the supremum over all such b yields

$$\mu_I(a) \leq \mu_I(a_1) + \mu_I(a_2) + \cdots .$$

Now suppose that $a = a_1 + a_2$ and that $b_1 \leq a_1$ and $b_2 \leq a_2$ with b_1, b_2 in I. Then $b_1 + b_2 \leq a$ with $b_1 + b_2 \in I$, so that

$$\mu_I(a) \geq \mu(b_1 + b_2) = \mu(b_1) + \mu(b_2).$$

Taking the supremum over all such b_1 and b_2 yields

$$\mu_I(a) \geq \mu_I(a_1) + \mu_I(a_2).$$

We have shown that μ_I is finitely additive and countably sub-additive.

Finally, suppose again that $a = a_1 + a_2 + \cdots$. Then

$$\mu_I(a) \geq \mu_I(a_1 + \cdots + a_n)$$

$$= \mu_I(a_1) + \cdots + \mu_I(a_n);$$

taking $n \to \infty$ yields countable super-additivity.

Q.E.D.

Certainly, if $I \subseteq J$, then $\mu_I \leq \mu_J$. Also, if $I = \{0\}$, then $\mu_I = 0$. Finally, $\mu_A = \mu$. Some of our work will rely on the observation that for each $a \in A$, the function $I \to \mu_I(a)$ behaves as a sort of generalised measure.

<u>1.12 Lemma</u>: Suppose that $\mu^{(1)}\ \mu^{(2)}\ \ldots$ are measures on a cardinal algebra A and that $I \subseteq A$ is an ideal. For each $c \in A$, there is some $c' \leq c$ such that $c' \in I$ and $\overset{(n)}{\mu_I}(c) = \overset{(n)}{\mu}(c')$ for each n.

<u>Note</u>: If also $\overset{(n)}{\mu_I}(c) < \infty$, and we write $c = c' + c''$, then $\overset{(n)}{\mu}(b) = 0$ whenever $b \leq c''$ with $b \in I$; otherwise put, we have $\overset{(n)}{\mu_I}(c'') = 0$.

<u>Proof</u>: Fix n. For each $k \geq 1$, there is some $a_k \in I$ such that $a_k \leq c$ and

$$\overset{(n)}{\mu}(a_k) \geq \overset{(n)}{\mu_I}(c) - 1/k \qquad \text{if } \overset{(n)}{\mu_I}(c) < \infty$$

or

$$\overset{(n)}{\mu}(a_k) \geq k \qquad \text{if } \overset{(n)}{\mu_I}(c) = \infty .$$

Use the interpolation property (lemma 1.3) to find $a \in A$ such that $a_k \leq a$ for each k and such that $a \leq c$ and $a \leq a_1 + a_2 + \cdots$. This last inequality implies that $a \varepsilon I$; the first shows that $\mu_I^{(n)}(c) = \mu^{(n)}(a)$.

We have shown that for each n, there is some $c_n \leq c$ such that $c_n \varepsilon I$ and $\mu_I^{(n)}(c) = \mu^{(n)}(c_n)$. Again apply interpolation (1.3) to find c' such that $c_n \leq c'$ for each n and such that $c' \leq c$ and $c' \leq c_1 + c_2 + \cdots$. We have $c' \varepsilon I$ and

$$\mu_I^{(n)}(c) \geq \mu^{(n)}(c') \geq \mu^{(n)}(c_n) = \mu_I^{(n)}(c_n),$$

with equality as desired.

Q.E.D.

<u>1.13 Measure extension lemma</u>: Let G be a subset of a cardinal algebra A such that $a \varepsilon G$ and $b \leq a$ together imply that $b \varepsilon G$. Suppose that $\mu : G \to [0,\infty]$ is a function such that $\mu(0) = 0$ and such that whenever $a = \sum a_n$ is an element of G, then $\mu(a) = \sum \mu(a_n)$. Then μ extends to a measure $\bar{\mu}$ defined on A.

<u>Indication</u>: One such desired extension may be defined by

$$\bar{\mu}(a) = \sup\{\sum \mu(b_n) : \sum b_n \leq a,\ b_n \varepsilon G\}.$$

Certainly, $\bar{\mu}(a) = \mu(a)$ for $a \varepsilon G$. The technique used to prove lemma 1.11 will show that $\bar{\mu}$ is a measure.

Q.E.D.

If μ is a measure on A, and $a \varepsilon A$, say that μ is <u>σ-finite at a</u> if $a = \sum a_n$ with $\mu(a_n) < \infty$ for each n. The collection of elements σ-finite for μ forms an ideal of A. If μ and ν are measures on A, say that ν is <u>absolutely</u> <u>countinuous</u> <u>with</u> <u>respect</u> <u>to</u> $\underline{\mu}$ (written $\nu << \mu$) if $\mu(a) = 0$ implies $\nu(a) = 0$ (equivalently, $\nu \leq \omega\mu$). The collection of all $\nu \in A^*$ such that $\nu << \mu$ forms an ideal of A^*: it is the principal ideal of A^* generated by μ.

Say that a cardinal algebra A is <u>countably</u> <u>generated</u> (c.g.) if there is a sequence $a_1\ a_2 \ldots$ in A such that every element of A is a sum of elements from this sequence. The cardinal algebra $A = [0, \infty]$ is c.g.— simply use rational elements as a generator.

<u>1.14 Lemma</u>: Let $N = \{0, 1, \ldots, \infty\}$ be the cardinal algebra of example 1.5 and let N^ω denote its countable product. A cardinal algebra A is c.g. if and only if there is a homomorphism mapping N^ω onto A.

<u>Proof</u>: Given $x \in N^\omega$, let $x(n)$ indicate its n-th co-ordinate. Define $x_1\ x_2 \ldots$ in N^ω by

$$x_k(n) = \begin{cases} 1 & n = k \\ 0 & n \neq k. \end{cases}$$

Then $\{x_1, x_2, \ldots\}$ generates N^ω. If $\varphi: N^\omega \to A$ is a homomorphism mapping N^ω onto A, then $\{\varphi(x_1), \varphi(x_2), \ldots\}$ generates A.

Conversely, suppose that $\{a_1, a_2, \ldots\}$ generates A. Define $\varphi : N^\omega \to A$ by

$$\varphi(n_1, n_2, \ldots) = \sum n_k a_k .$$

Then φ is a homomorphism onto A.

Q.E.D.

This result immediately implies the following corollary.

1.15 Lemma: Suppose $A = A_1 \times A_2 \times \cdots$ is a product of the cardinal algebras A_n. Then A is c.g. if and only if each A_n is c.g..

I do not know the answer to the following problem. Its resolution in the positive would imply that A^* is c.g. whenever A is c.g..

1.16 Question: Is every sub-algebra of a c.g. cardinal algebra again c.g.?

1.17 Question: Is A^* c.g. whenever A is c.g.?

A cardinal algebra A we call separable if every subset of A well-ordered by $\leq$ is countable. Clearly, every sub-algebra of a separable cardinal algebra is again separable. Also, a countable product of separable algebras is again separable. Thus $[0, \infty]$ and $[0, \infty]^\omega$ are separable. Indeed, every c.g. cardinal algebra is separable. But $A = \omega_1 \cup \{\omega_1\}$ under the operation

$$\alpha_1 + \alpha_2 + \cdots = \inf(\alpha_1, \alpha_2, \ldots)$$

is a separable cardinal algebra, yet is not c.g..

1.18 Lemma: Let A be a separable cardinal algebra. Then

1) A is a complete, distributive lattice;

2) if $B \subseteq A$, then there is some countable $C \subseteq B$ with

sup C = sup B.

Note: In particular, this means that A has a largest element.

Indication: This may be pieced together from the following points in Tarski's book [15]: Theorem 3.35, Theorem 3.4, Theorem 3.21.

1.19 Lemma: If A is a c.g. cardinal algebra, then A^* is separable.

Proof: By lemma 1.14, there is a homomorphism φ mapping N^ω onto A. Define $\varphi^* : A^* \to [0, \infty]^\omega$ by

$$\varphi^*(\mu)(n) = \mu(\varphi(x_n)),$$

where x_n is as in the proof of lemma 1.14. Then φ^* is a homomorphism and is one-one. Thus A^* is isomorphic with a cardinal sub-algebra of $[0, \infty]^\omega$. It follows that A^* is separable.

Q.E.D.

We now turn our attention to the representation theory for distributive lattices developed by Priestley [7] and improving on earlier work of Stone [13]. Let $\leq$ be a partial order on a set S. A set $U \subseteq S$ is increasing if $u \in U$ and $v \geq u$ together imply $v \in U$. If S is also a topological space, one calls $(S, \leq)$ totally order disconnected if whenever $a \nleq b$ in S implies that there is some clopen increasing $U \subseteq S$ with $a \in U$ and $b \in S - U$. Clearly, this implies that S is Hausdorff. Also, the clopen increasing subsets of S are closed under finite union and intersection and so constitute a lattice, the so-called <u>dual lattice</u> of $(S, \leq)$.

<u>1.20 Priestley's Theorem</u>: Let $(L, \leq)$ be a distributive lattice with top and bottom (1 and 0). Then there is a totally order disconnected compact space $(S, \leq)$ such that $(L, \leq)$ is isomorphic to the dual lattice of $(S, \leq)$.

<u>Indication</u>: This is Theorem 1 in [7].

The space S may be constructed as the set of proper prime ideals of L with the desired isomorphism sending $a \in L$ to the set U_a of all prime ideals of L not containing a. The order on S is given by $s \leq s'$ in case the prime ideal s contains the prime ideal s'. Call S the <u>Stone-Priestly dual space</u> of L. Priestley's Theorem has the advantage over Stone's original in that it produces a Hausdorff space (not merely a T_0 space).

Given a cardinal algebra A, the set $L = \mathrm{Idl}(A)$ of all cardinal algebra ideals of A forms a distributive lattice, as mentioned before. Thus, we may construct $(S, \leq)$, the Stone-Priestley dual of $L = \mathrm{Idl}(A)$. Each ideal $I \in \mathrm{Idl}(A)$ is thus identified with a clopen increasing set $\tilde{I} \subseteq S$ under the Priestley isomorphism. We shall make extensive use of the correspondence $I \to \tilde{I}$ in the sequel.

We shall also require the use of a very nice result of Tarski regarding the extension of finitely additive measures in a very general setting. Let $(T, +)$ be a commutative monoid with neutral element 0: we mean that $+$ is an associative, commutative operation on T and that $0 \in T$ is an identity element for $+$. Call a function $g : T \to [0, \infty]$ a <u>mass-function</u> on T if g is a monoid-morphism: $g(x + y) = g(x) + g(y)$ whenever $x, y \in T$. We wish to know when a function f defined on a subset U of T extends to a mass-function on T. Define a relation $\leq$ on T by setting $x \leq y$ if there is some $z \in T$ with $y = x + z$. (Note that $\leq$ might not be a partial order on T.)

<u>1.21 Tarski extension lemma</u>: In the context just outlined, suppose that $f : U \to [0, \infty)$ is a function on a set $U \subseteq T$. Then f extends to a mass-function on T if and only if the following condition is satisfied:

(*) Whenever $x_1, \ldots, x_n, x_{n+1}, \ldots, x_m$ are elements of U such that $x_1 + \cdots + x_n \leq x_{n+1} + \cdots + x_m$ in T, then $f(x_1) + \cdots + f(x_n) \leq f(x_{n+1}) + \cdots + f(x_m)$.

<u>Indication</u>: This follows from Satz 1.55 in [14], keeping Tarski's Definition 1.11 in view.

To conclude this section, we state a couple of basic results from elementary measure theory.

<u>1.22 Lemma</u>: Let C be the algebra of clopen subsets of a compact Hausdorff space S. Every finitely additive measure $m : C \to [0, \infty)$ extends uniquely to a countably additive measure on the σ-field generated by C.

<u>Note</u>: If S is a totally disconnected compact space, then this is the Baire σ-field of S, which separates points.

Indication: Since a disjoint union of infinitely many non-empty open sets cannot be compact, m is already countably additive on $\mathcal{C}$. The Carathéodory extension applies. See [8; p.295].

1.23 Lemma: Suppose that $(S, \mathcal{B})$ is a measurable space and that B_t is a family of sets in $\mathcal{B}$ indexed by $t \geq 0$ with $B_0 = S$ and such that $B_t \supseteq B_{t'}$ whenever $t \leq t'$. Then there is a $\mathcal{B}$-measurable function $f : S \to [0, \infty]$ such that

$$f(s) \geq t \qquad \text{for} \qquad s \in B_t$$

$$f(s) \leq t \qquad \text{for} \qquad s \in S - B_t \,.$$

Indication: Such a function f may be defined by the formula

$$f(s) = \sup\{t : s \in B_t \,,\ t \ \text{rational}\}.$$

Compare lemma 11.2.9 in [8].

Lastly, we mention our use of the symbol 1_B to denote the indicator function of a set B:

$$1_B(x) = \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \in B \,. \end{cases}$$

§2. Integration

At the outset, it is not clear how to develop a natural integration theory for cardinal algebras. The measures are defined simply enough, but where are the functions to be integrated? A study of the classical proof of the Radon-Nikodym theorem provided the motivation for our approach, and indeed, in section 4 we present a type of Radon-Nikodym result for measures on cardinal algebras.

In this section, we show how each measure μ on a cardinal algebra A induces a family of (ordinary) measures on the Stone-Priestley dual S of the lattice $\mathrm{Idl}(A)$. The functions to be integrated are then functions on S.

2.1 Proposition: Let μ be a measure on a cardinal algebra A and suppose that S is the Stone-Priestley dual of the distributive lattice $\mathrm{Idl}(A)$. Let $C \subseteq A$ be the ideal of all elements σ-finite for μ. Then there is a unique function $c \to \varphi_c$ defined on C such that

1) each φ_c is a Baire measure on S;

2) whenever $I \subseteq A$ is an ideal, then $\varphi_c(\tilde{I}) = \mu_I(c)$;

3) the function $c \to \varphi_c$ is a cardinal algebra homomorphism defined on C.

Demonstration: First, we suppose that $\mu(c) < \infty$. Define G to be the set of all functions on S of the form $1_{K_1} + 1_{K_2} + \cdots + 1_{K_n}$, where n is a positive integer and $K_1 \ldots K_n$ are clopen sets. Clearly, G is a partially ordered monoid; also, if g and g' are elements of G, then $g \leq g'$ point-wise on S if and only if there is some g'' in G such that $g' = g + g''$ (G is a "divisibility monoid"). Let H be the set of all functions on S of the form $1_{\tilde{I}}$, where I is an ideal of A. We now define a function f on H by setting $f(1_{\tilde{I}}) = \mu_I(c)$. Since the correspondence $I \to \tilde{I}$ is one-one, f is well-defined on H.

It is our wish to extend the domain of f to all of G by means of Tarski's Theorem (lemma 1.21). To this end, we make the

Claim: If $I_1, \ldots, I_n, I_{n+1}, \ldots, I_m$ are ideals of A, and

$$(*) \quad 1_{\tilde{I}_1}(s) + \cdots + 1_{\tilde{I}_n}(s) \leq 1_{\tilde{I}_{n+1}}(s) + \cdots 1_{\tilde{I}_m}(s)$$

for each $s \in S$, then

$$(**) \quad \mu_{I_1}(c) + \cdots + \mu_{I_n}(c) \leq \mu_{I_{n+1}}(c) + \cdots + \mu_{I_m}(c).$$

Proof of claim: Put $\mathcal{I} = \{1, \ldots, n, n+1, \ldots, m\}$. For each $k = 1, \ldots, m$, we may write $c = c_k + c_k'$ with $c_k \in I_k$ and $\mu_{I_k}(c) = \mu(c_k)$ as in lemma 1.12. For each $b \in I_k$ with $b \leq c_k'$, we have $\mu(b) = 0$. Using refinement, we may find, for each $M \subseteq \mathcal{I}$, elements $c(M)$ such that

$$c = \sum \{c(M) : M \subseteq \mathcal{I}\}$$

and such that for each $k = 1, \ldots, m$,

$$c_k = \sum \{c(M) : k \in M \subseteq \mathcal{I}\}$$

$$c_k' = \sum \{c(M) : k \notin M \subseteq \mathcal{I}\}.$$

It suffices to prove the inequality (**) for each element $c(M)$.

Next, let I_0 be the ideal of all $a \varepsilon A$ with $\mu(a) = 0$ and let $I(c(M))$ be the principal ideal generated by $c(M)$. There are two cases to consider. If $\mu(c(M)) = 0$, inequality (**) is trivial. In the other case, $\mu(c(M)) > 0$ and $I(c(M)) \nsubseteq I_0$. Choose $s \varepsilon \tilde{I}(c(M) - \tilde{I}_0$. We have

$$I(c(M)) \subseteq I_K \qquad \text{if } k \varepsilon M$$

$$I(c(M) \cap I_K \subseteq I_0 \qquad \text{if } k \varepsilon I - M$$

so that

$$\mu_{I_k}(c(M)) = \mu(c(M)) \qquad \text{if } k \varepsilon M$$

$$\mu_{I_k}(c(M)) = 0 \qquad \text{if } k \varepsilon I - M.$$

Put $M = N \cup P$, where $N \subseteq \{1, \ldots, n\}$ and $P \subseteq \{n+1, \ldots, m\}$. Inequality (*) applied at the chosen point s yields $\text{card}(N) \leq \text{card}(P)$. Inequality (**) follows. The claim is proved.

Tarski's Extension Theroem (1.21) now applies: the function f extends to a homomorphism $\overline{f}$ of G into $[0, \infty]$. The mapping $K \to f(1_K)$ is therefore a finitely additive measure defined on the clopen algebra of S. By lemma 1.22, this mapping extends to a Baire measure φ_c on S. For each ideal $I \subseteq A$, we have

$$(***) \qquad \varphi_c(\tilde{I}) = f(1_{\tilde{I}}) = \mu_I(c) .$$

As noted in [7], the collection of all such $\tilde{I}$ generates the clopen algebra and hence the Baire σ-field. Since the collection is closed under finite intersections, Dynkin's Lemma [3: 1.6.2] implies that φ_c is the unique Baire measure on S satisfying (***). Furthermore, the same point shows that if $c = \sum c_n$, then $\varphi_c = \sum \varphi_{c_n}$.

We now turn to the case where c is only σ-finite for μ. Write $c = \sum c_n$ with $\mu(c_n) < \infty$ for each n and define $\varphi_c = \sum \varphi_{c_n}$. We show that φ_c is well-defined: if $c = \sum c'_n = \sum c_n$ with $\mu(c'_n) < \infty$, then the refinement property (lemma 1.1) implies that

$$c_n = \sum_m d_{nm} \qquad c'_m = \sum_n d_{nm}$$

for certain elements d_{nm}. Then, by our previous remarks,

$$\sum_n \varphi_{c_n} = \sum_n \sum_m \varphi_{d_{nm}} = \sum_m \sum_n \varphi_{d_{nm}}$$

$$= \sum_m \varphi_{c'_m}$$

as desired. It then becomes obvious that $c \to \varphi_c$ is a cardinal algebra homomorphism on C.

Q.E.D.

Let μ be a measure on a cardinal algebra A and let S be the Stone-Priestley dual of $\mathrm{Idl}(A)$. Suppose that μ is σ-finite at $c \in A$ and that f is an extended real-valued function on S. We say that f is <u>μ-integrable over c</u> if f is φ_c-integrable on S (in the usual sense of Lebesgue). Here, φ_c is the unique measure on S guaranteed to exist by proposition 2.1. In this definition, we shall mean that f is Baire-measurable on S; however, the definition could be extended to Borel functions by forming the regular Borel extension of φ_c. In any case, we may then define the <u>integral</u> of f over c by

$$\int_c f d\mu = \int_S f d\varphi_c .$$

In view of the definition of φ_c, one may more descriptively write

$$\int_c f d\mu = \int_S f(s)\mu_{ds}(c) .$$

It is also clear that this integral may be defined under the assumption that f is non-negative and measurable. We have

<u>2.2 Lemma</u>: In the same context, suppose that the integrals

$$\int_c f d\mu \qquad \int_c f' d\mu \qquad \int_c f'' d\mu$$

are defined. Suppose that $c = \sum c_n$ and $\mu = \sum \mu_n$ and $f = f' + f''$. Then

1) $\displaystyle\int_c f d\mu = \sum_n \int_{c_n} f d\mu$

2) $\displaystyle\int_c f d\mu = \sum_n \int_c f d\mu_n$

3) $\displaystyle\int_c f d\mu = \int_c f' d\mu + \int_c f'' d\mu$.

<u>Indication</u>: Part 1 follows from the countable additivity of the map

$c \to \varphi_c$. For part 2, we need verify only the

Claim: For each ideal $I \subseteq A$, we have the equality

$$\mu_I(c) = \sum \mu_{n,I}(c) \ .$$

Proof of claim: We check this when $\mu(c) < \infty$ and rely on σ-finiteness. By lemma 1.12, there is some $c' \in I$ such that $c' \leq c$ and

$$\mu_I(c) = \mu(c') \qquad \mu_{n,I}(c) = \mu_n(c') \ .$$

The claim follows easily.

Part 3 follows from elementary properties of ordinary Lebesgue integration.

Q.E.D.

In view of this result, the integral given by

$$\int_c f d\mu$$

may be considered as a tri-linear form, countably additive in each of the arguments c, f and μ.

2.3 Lemma: In the same context, suppose that f is non-negative and measurable on S. Then

$$\int_c f d\mu = \int_0^\infty \varphi_c \{s : f(s) \geq t\} \, dt$$

$$= \int_0^\infty \varphi_c \{s : f(s) > t\}\, dt \; .$$

Indication: In the case where $\mu(c) < \infty$, this is a standard application of Fubini's Theorem for Lebesgue integrals (vide e.g. [3: page 163]). For c σ-finite, use additivity (lemma 2.2.1) and the ordinary Monotone Convergence Theorem.

There are several ways to define the integral of f over elements c at which μ is not σ-finite. Since our applications apply only to the σ-finite case, we refrain from making a definition in this situation; future applications will perhaps make a particular formulation appear most natural.

We now formulate versions of the classical convergence theorems of Fatou, Beppo Levi and Lebesgue. Each follows immediately from the corresponding classical result applied to f and the measure φ_c.

2.4 Proposition: Let μ be a measure on a cardinal algebra A and suppose that μ is σ-finite at c. Let φ_c be the measure induced by μ on the Stone-Priestley dual space S. If $f_1\ f_2\ \ldots$ is a sequence of non-negative measurable functions on S such that $f_n \to f$ φ_c - almost everywhere, then

$$\int_c f d\mu \leq \liminf_n \int_c f_n \, d\mu \; .$$

2.5 Proposition: In the same context, suppose that $f_n(s)$ increases to $f(s)$ as $n \to \infty$ for φ_c-almost all $s \in S$. Then

$$\int_c f d\mu = \lim_n \int_c f_n \, d\mu .$$

<u>2.6 Proposition</u>: In the same context, suppose that g and $f_1\ f_2 \ldots$ are measurable, extended real-valued functions on S converging φ_c - almost everywhere to f. Suppose that $|f_n| \leq |g|$ a.e. on S and that g is μ-integrable over c. Then $f_1\ f_2 \ldots$ and f are μ-integrable over c, and

$$\int_c f d\mu = \lim_n \int_c f_n \, d\mu .$$

§3. Signed measures

Aside from being interesting in their own right, signed measures play an important role in the usual proof of the Radon-Nikodym Theorem. In this regard, what holds for ordinary measures applies equally to the case of measures on cardinal algebras; the results of this section will be used to prove a theorem of Radon-Nikodym type in the next section. Unfortunately, signed measures cannot be defined on an entire cardinal algebra: one runs into the usual problem of collision of infinities $\infty - \infty$. Thus, signed measures will be defined on semi-ideals of a cardinal algebra.

Let F be a semi-ideal in a cardinal algebra A. A function $\mu : F \to [-\infty, \infty]$ is a <u>signed measure</u> on F if

1) μ assumes at most one of the values $+\infty$ and $-\infty$;

2) $\mu(0) = 0$;

3) whenever $a \in F$ and $a = a_0 + a_1 + \cdots$, then $\mu(a) = \mu(a_0) + \mu(a_1) + \cdots$, with absolute convergence if $|\mu(a)| < \infty$.

An element u of F is <u>positive</u> [resp. <u>negative</u>] with respect to μ if $\mu(v) \geq 0$ [resp. $\mu(v) \leq 0$] for each $v \leq u$. Say that u is <u>null</u> if it is both positive and negative.

<u>3.1 Lemma</u>: Let u be positive [resp. negative] with respect to the signed measure μ on F. Then each $v \leq u$ is positive [resp. negative]. If $v_0\ v_1 \ldots$ are positive [resp. negative], and $v = v_0 + v_1 + \cdots$ belongs to F, then v is positive [resp. negative].

Indication: The first assertion is obvious. The second follows from lemma 1.2.

3.2 Lemma: Let μ be a signed measure on F and suppose that $b \in F$ is such that $0 < \mu(b) < \infty$. Then there is some positive $c \leq b$ with $\mu(c) > 0$.

Proof: If b is positive, there is nothing to prove. If not, let n_0 be the smallest positive integer for which there is some $c_0 \leq b$ with $\mu(c_0) < - 1/n_0$. We proceed inductively: suppose that $c_0\ c_1\ \ldots\ c_{k-1}$ have been defined so that $c_0 + c_1 + \cdots + c_{k-1} \leq b$. Let n_k be the smallest positive integer for which there is some c_k such that

$$c_0 + c_1 + \cdots + c_k \leq b \quad \text{and} \quad \mu(c_k) < - 1/n_k.$$

Let c be an element such that

$$b = c + c_0 + c_1 + \cdots .$$

Then

$$\mu(b) = \mu(c) + \sum \mu(c_k)$$

with the infinite series absolutely convergent. It follows that $\sum 1/n_k$ converges and therefore that $n_k \to \infty$ as $k \to \infty$. Also, $\mu(c) > 0$.

We now show that c is a positive element. Given $\varepsilon > 0$, choose k large enough that

$$\frac{1}{n_k - 1} < \varepsilon .$$

Suppose that $u \le c$ with $\mu(u) < -\varepsilon < -(n_k - 1)^{-1}$. This contradicts the definition of n_k and c_k. Thus $u \le c$ implies $\mu(u) \ge -\varepsilon$. Since $\varepsilon > 0$ was arbitrary, c is indeed positive.

Q.E.D.

We now formulate and prove a version of the Hahn decomposition theorem for cardinal algebra signed measures.

3.3 Proposition: Let μ be a signed measure on a semi-ideal F. For each $a \in F$, we may write $a = p + q$, where p is positive and q negative.

If $a = p' + q'$ is another such decomposition, then there is a positive p'' and negative q'' such that

$$p = p'' + r \qquad q = q'' + s$$

$$p' = p'' + r' \qquad q' = q'' + s' ,$$

where r r' s s' are null elements.

Demonstration: Without loss of generality, we assume that $\mu(b) < \infty$ for $b \in F$. Define

$$y = \sup\{\mu(c) : c \le a, \ c \text{ positive}\}.$$

Choose elements c_n and d_n such that $a = c_n + d_n$, c_n is positive, and $\mu(c_n) > y - 1/n$. Since

$$a = c_1 + d_1 = c_2 + d_2,$$

we may use refinement to write

$$c_1 = e_{11} + e_{12} \qquad c_2 = e_{11} + e_{21}$$

$$d_1 = e_{21} + c_{22} \qquad d_2 = e_{12} + e_{22} .$$

Put $p_1 = c_1$ and $p_2 = c_1 + e_{21} = e_{11} + e_{12} + e_{21}$. Then p_1 and p_2 are positive, $p_1 \leq p_2$, and $\mu(p_2) \geq \mu(c_2)$. Continuing in this wise, we may produce a sequence $p_1\ p_2\ p_3 \ldots$ of increasing positive elements with

$$\mu(p_n) \geq \mu(c_n) \geq y - 1/n .$$

Put $p = \sup p_n$. Then p is positive, and $\mu(p) = y$.

Choose q so that $a = p + q$. If $u \leq q$ with $\mu(u) > 0$, then by lemma 3.2, there is some positive $v \leq u$ with $\mu(v) > 0$. Then $p + v \leq a$ is positive, so that $\mu(p + v) \leq y$. But this implies that

$$\mu(p) < \mu(p) + \mu(v) = \mu(p + v) \leq y = \mu(p),$$

a contradiction. It follows that q is negative.

The last statement of the proposition follows quickly from the refinement property.

Q.E.D.

Continuing with this train of thought, suppose that $b \leq a = p + q$. Then (lemma 1.2), $b = u + v$ with $u \leq p$ and $v \leq q$. Set

$$\mu^+(b) = \mu(u) \qquad \mu^-(b) = -\mu(v) .$$

A refinement argument shows that μ^+ and μ^- extend to measures on the ideal generated by F. By lemma 1.13 μ^+ and μ^- extend to measures on the algebra A. Then μ is the restriction of $\mu^+ - \mu^-$ to F. This is the "Jordan decomposition" of μ.

In the usual Hahn decomposition for measures defined on σ-fields, the underlying space is partitioned into disjoint sets—one positive, one negative. The notion of disjointness is not so easy to deal with in the context of cardinal algebras. The following lemma, which shall be used in the next section, attempts to master this difficulty.

3.4 Lemma: Let μ be a measure on a cardinal algebra A and suppose that ν is a signed measure on a semi-ideal $F \subseteq A$. Given $a \in F$ with $\mu(a) < \infty$, we may write $a = p + q$ with

1) p positive for ν ;

2) q negative for ν ;

3) $\mu(b) = 0$ whenever $b \leq q$ is positive for ν.

Proof: Let $a = p' + q'$ be a Hahn decomposition of a as in proposition 3.3. Define

$$s = \sup\{\mu(b) \ ; \ b \leq q' \text{ is positive for } \nu\}$$

and take b_n positive for ν with $b_n \leq q'$ and $\lim\mu(b_n) = s$. Use interpolation (lemma 1.3) to find c with $b_n \leq c \leq \sum b_m$ for each n and such that $c \leq q'$. Then c is positive for ν, and $\mu(c) = s$. Find q such that $q' = q + c$ and put $p = p' + c$. Then $a = p + q$ is a Hahn decomposition with the desired characteristics.

Q.E.D.

§4. Radon-Nikodym theorems

We are now ready for a statement and proof of the Radon-Nikodym theorem for cardinal algebras. Recall that σ-finiteness is a natural hypothesis for such a result.

4.1 Proposition: Let μ and ν be measures on a cardinal algebra A with $\nu \ll \mu$. Let S be the Stone-Priestley dual space of $\mathrm{Idl}(A)$. There is a Baire-measurable function $f : S \to [0, \infty]$ such that

$$\nu(c) = \int_c f \, d\mu$$

whenever μ is σ-finite at c.

Demonstration: Let F be the semi-ideal of all $a \in A$ with $\mu(a) < \infty$. For each $t \in [0, \infty)$, we see that $\nu - t\mu$ is a signed measure on F. Let I_t be the ideal generated by the elements of F that are positive for $\nu - t\mu$. Clearly, $t \leq t'$ implies that $I_t \supseteq I_{t'}$ and therefore $\tilde{I}_t \supseteq \tilde{I}_{t'}$ as subsets of S. Lemma 1.23 asserts the existence of a Baire-measurable function $f : S \to [0, \infty]$ such that

$$f(s) \geq t \quad \text{for} \quad s \in \tilde{I}_t \text{ , and}$$

$$f(s) \leq t \quad \text{for} \quad s \in S - \tilde{I}_t \text{ .}$$

(In fact, one can check that f is lower semi-continuous on S.)

Now fix a positive integer N and assume that $\mu(c) < \infty$. For each $k \geq 0$, use lemma 3.4 to write $c = p_k + q_k$ with

1) p_k positive for $\nu - \frac{k}{N}\mu$;

2) q_k negative for $\nu - \frac{k}{N}\mu$;

3) $\mu(b) = 0$ whenever $b \leq q_k$ is positive for $\nu - \frac{k}{N}\mu$.

We insist that $p_0 = c$ and $q_0 = 0$. We now define elements $u_k\ v_k\ c_k$ by inductive process. Put $u_0 = p_0 = c$ and $v_0 = q_0 = 0$ and suppose that u_k and v_k have been defined so that $c = u_k + v_k$. Use refinement to write

$$u_k = d_{11} + d_{12} \qquad p_{k+1} = d_{11} + d_{21}$$

$$v_k = d_{21} + d_{22} \qquad q_{k+1} = d_{12} + d_{22} .$$

Put

$$u_{k+1} = d_{11} \qquad v_{k+1} = d_{12} + d_{21} + d_{22} \qquad c_k = d_{12} .$$

We note that

$$u_k = u_{k+1} + c_k$$

$$c_k \leq u_k \leq p_k$$

$$c_k \leq q_{k+1}$$

for each $k = 0, 1, 2, \dots$.

Applying the remainder postulate to the relation $u_k = u_{k+1} + c_k$ yields the existence of an element c_∞ such that, for each $k \geq 0$,

$$u_k = c_\infty + c_k + c_{k+1} + \cdots ;$$

in particular, $c_\infty \leq u_k$ for each k, and

$$c = c_\infty + c_0 + c_1 + \cdots .$$

Claim 1: For $k = 0, 1, \ldots$, we have the inequality

$$\frac{k}{N}\mu(c_k) \leq \int_{c_k} f \, d\mu .$$

Proof of claim: As noted previously, $c_k \leq u_k \leq p_k$ and $p_k \in I_{k/N}$. Thus $c_k \in I_{k/N}$. We calculate:

$$\int_{c_k} f \, d\mu = \int_S f(s) \, d\varphi_{c_k}(s)$$

$$\geq \int_{\tilde{I}_{k/N}} f(s) \, d\varphi_{c_k}(s)$$

$$\geq \frac{k}{N} \mu_{I_{k/N}}(c_k) = \frac{k}{N} \mu(c_k) ,$$

as desired.

Claim 2: For $k = 0, 1, \ldots$, we have the inequality

$$\int_{c_k} f \, d\mu \leq \frac{k+1}{N} \mu(c_k).$$

Proof of claim: We use lemma 2.3 to write

$$\int_{c_k} f\,d\mu = \int_S f(s)\,d\varphi_{c_k}(s) = \int_0^\infty \varphi_{c_k}\{s : f(s) > t\}dt$$

$$\leq \int_0^\infty \varphi_{c_k}(\tilde{I}_t)dt = \int_0^\infty \mu_{I_t}(c_k)dt$$

$$= \int_0^{\frac{k+1}{N}} \mu_{I_t}(c_k)dt + \int_{\frac{k+1}{N}}^\infty \mu_{I_t}(c_k)dt .$$

Since $c_k \leq q_{k+1}$, we see that $\mu(b) = 0$ whenever $b \leq c_k$ is positive for

$$\nu - \frac{k+1}{N}\mu .$$

(We have used lemma 3.4!) It follows that $\mu_J(c_k) = 0$ for $J = I_{\frac{k+1}{N}}$.
For $t \geq (k+1)/N$, we have

$$\mu_{I_t}(c_k) \leq \mu_J(c_k) = 0,$$

so that

$$\int_{c_k} f\,d\mu \leq \int_0^{\frac{k+1}{N}} \mu_{I_t}(c_k)dt \leq \int_0^{\frac{k+1}{N}} \mu(c_k)dt = \frac{k+1}{N}\mu(c_k)$$

as desired.

Since $c_k \leq p_k$ and $c_k \leq q_{k+1}$, we have the inequality

$$\frac{k}{N}\mu(c_k) \leq \nu(c_k) \leq \frac{k+1}{N}\mu(c_k) .$$

Combining this with claims 1 and 2 yields

$$\nu(c_k) - \frac{1}{N}\mu(c_k) \le \int_{c_k} f\, d\mu \le \nu(c_k) + \frac{1}{N}\nu(c_k)$$

for $k = 0, 1, \dots$. For c_∞, we have $c_\infty \le u_k \le p_k$ for each k, so that

$$(\nu - \tfrac{k}{N}\mu)(c_\infty) \ge 0$$

for each k. This forces either $\mu(c_\infty) = 0$, or $\mu(c_\infty) > 0$ and $\nu(c_\infty) = \infty$. In the former case, we use absolute continuity to see that

$$\nu(c_\infty) = 0 = \int_{c_\infty} f\, d\mu \ ;$$

in the latter, note that $c_\infty \le p_k$ $(k \ge 0)$ implies that $c_\infty \in I_t$ for each $t \ge 0$. Then we have

$$\int_{c_\infty} f\, d\mu = \int_S f(s)\, d\varphi_{c_\infty}(s)$$

$$= \int_0^\infty \varphi_{c_\infty}(f^{-1}[t, \infty])dt \ge \int_0^\infty \mu_{I_t}(c_\infty)dt$$

$$= \int_0^\infty \mu(c_\infty)dt = \infty = \nu(c_\infty) \ .$$

Summing over $k = 0, 1, \dots, \infty$ yields

$$\nu(c) - \frac{1}{N}\mu(c) \le \int_c f\, d\mu \le \nu(c) + \frac{1}{N}\mu(c) \ .$$

Since $\mu(c) < \infty$ and N is arbitrary, we have

$$\nu(c) = \int_c f \, d\mu$$

as desired. The same result follows easily for each c at which μ is σ-finite (using additivity from lemma 2.2.1).

Q.E.D.

If f is a function guaranteed to exist by proposition 4.1, we call f a Radon-Nikodym derivative of ν with respect to μ and write

$$f = \frac{d\nu}{d\mu} .$$

The usual calculus of such derivatives may be developed. We offer the following instance:

4.2 Proposition: Let μ and ν be measures on a cardinal algebra A and suppose that $\nu \ll \mu$. Let f be a Radon-Nikodym derivative guaranteed to exist by proposition 4.1. Suppose that μ and ν are σ-finite at $c \in A$. If g is a function on S which is either

1) non-negative and measurable, or

2) ν-integrable over c,

then

$$(*) \qquad \int_c g d\nu = \int_c g f d\mu .$$

<u>Demonstration</u>: Using σ-finiteness for μ and ν together with additivity (lemma 2.2.1) and refinement (lemma 1.1), it suffices to verify (*) for the case where $\nu(c) < \infty$ and $\mu(c) < \infty$. We assume first that $g = 1_{\tilde{I}}$ for some ideal $I \subseteq A$. Use lemma 1.12 to write $c = c' + c''$ with $c' \in I$ and $\nu_I(c) = \nu(c')$ and also $\mu_I(c) = \mu(c')$. Let $a \to \varphi_a$ be the mapping guaranteed to exist (relative to μ) by proposition 2.1. Then

$$\varphi_{c'}(\tilde{I}) = \mu_I(c') = \mu(c') = \varphi_{c'}(S)$$

$$\varphi_{c''}(\tilde{I}) = \mu_I(c'') = 0 ,$$

so that

$$\int_c 1_{\tilde{I}}\, d\nu = \nu_I(c) = \nu(c')$$

$$= \int_{c'} f\, d\mu = \int_S f\, d\varphi_{c'}$$

$$= \int_{\tilde{I}} f\, d\varphi_{c'} = \int_{\tilde{I}} f\, d\varphi_{c'} + \int_{\tilde{I}} f\, d\varphi_{c''}$$

$$= \int_{\tilde{I}} f\, d\varphi_c = \int_c 1_{\tilde{I}} f\, d\mu ,$$

as desired. Now both of the set functions

$$B \to \int_c 1_B\, d\nu \qquad\qquad B \to \int_c 1_B f\, d\mu$$

define finite Baire measures on S. Since these measures agree on sets of the form $\tilde{I}$, they must be equal (use Dynkin's lemma [3: p.163]).

Linearity (lemma 2.23) implies the result for g a Baire-measurable simple function. Taking limits (proposition 2.6) finishes the work.

Q.E.D.

§5. Application: integral representations for A^{**}

Let A be a cardinal algebra and let A^{**} be the second dual of A. As in most duality theories, there is a canonical homomorphism $T : A \to A^{**}$ defined by putting $T(a)(\mu) = \mu(a)$. However, as pointed out in [12], A^* need not separate the points of A, so that the canonical representation T might not be injective; also T is often not a surjection. The version of the Radon-Nikodym presented in the previous section allows us to compensate for this in some ways. First, we note the following

<u>5.1 Lemma</u>: Let A be a countably generated cardinal algebra and let T be the canonical mapping from A to A^{**}. For each U in A^{**}, there is some element $a \in A$ such that U and $T(a)$ are equivalent (mutually absolutely continuous) as cardinal algebra measures on A^*.

<u>Proof</u>: Using lemmas 1.19 and 1.18.1, we may put $\nu = \sup\{\mu : U(\mu) = 0\}$. From lemma 1.18.2 follows $U(\nu) = 0$. And by the same token, we put $a = \sup\{b : \nu(b) = 0\}$ and note that $\nu(a) = 0$. If now $U(\mu) = 0$, then $\mu \leq \nu$, and $T(a)(\mu) = \mu(a) \leq \nu(a) = 0$. This shows that $T(a) << U$. Next, suppose $T(a)(\mu) = \mu(a) = 0$. Then $\mu << \nu$ and $\mu \leq \omega\nu$, so that $U(\mu) \leq U(\omega\nu) = \omega U(\nu) = 0$. Thus $U << T(a)$.

Q.E.D.

In the context of this lemma, we would like to be able to conclude that $U = T(a)$, i.e. that $U(\mu) = \mu(a)$. This would establish an isomorphism of A with A^{**}. However, although such an isomorphism obtains for some cardinal algebras ($A = [0, \infty]$ or countable products thereof), it fails for others $(A = \{0, 1, \ldots, \infty\})$. Despite the absence of such a general reflexivity result, we can at least say the following: if A is countably generated, and $\mu(a) < \infty$, then there is a family of measures ν_t $(t \geq 0)$ majorised by μ such that

$$U(\mu) = \int_0^\infty \nu_t(a)dt \ .$$

Although U may not be evaluation at a, it is a sort of average of evaluations of the ν_t. More can be said of the integral kernel ν_t, and we presently make a formal statement of this result.

<u>5.2 Proposition</u>: Let A be a countably generated cardinal algebra and let U be a measure on A^*. For each $t \geq 0$, there is a function $\nu_t : A \times A^* \to [0, \infty]$ such that

$$\nu_t(\sum b_n, \mu) = \sum \nu_t(b_n, \mu)$$

$$\nu_t(b, \sum \mu_n) = \sum \nu_t(b, \mu_n) \ .$$

Also,

$$\nu_t(b, \mu) \leq \nu_{t'}(b, \mu) \leq \mu(b)$$

whenever $t \geq t'$. There is some $a \in A$ such that

$$U(\mu) = \int_0^\infty \nu_t(a, \mu)dt$$

whenever $\mu(a) < \infty$.

Demonstration: Let a be the element guaranteed to exist by lemma 5.1. Then U and $T(a)$ are equivalent measures. Let f be a Radon-Nikodym derivative ($f = dU/dT(a)$) as in proposition 4.1. Inspecting the proof of that proposition, we find that if $\mu(a) < \infty$, then

$$U(\mu) = \int_{\mu} f \, dT(a)$$

$$= \int_{0}^{\infty} \varphi_{\mu}\{s : f(s) > t\}dt$$

$$= \int_{0}^{\infty} T(a)_{I_t}(\mu)dt \ .$$

Here, $\mu \to \varphi_{\mu}$ is the homomorphism for the measure $T(a)$ from proposition 2.1, and $I_t \subseteq A^*$ is the ideal generated by the positive elements for $U - tT(a)$. Define

$$\nu_t(\cdot, \mu) = \sup\{p \in A^* : p \leq \mu \ \text{ and } \ p \in I_t\}.$$

Then ν_t has the properties advertised, and

$$U(\mu) = \int_{0}^{\infty} T(a)_{I_t}(\mu)dt$$

$$= \int_{0}^{\infty} T(a)(\nu_t(\cdot, \mu))dt$$

$$= \int_{0}^{\infty} \nu_t(a, \mu)dt$$

as desired.

Q.E.D.

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