RESEARCH ARTICLES *Real Analysis Exchange Vol 16 (1990-91)*

Rana Barua, Stat-Math Division, Indian Statistical Institute, Calcutta 700035,India.

ON THE BOREL HIERARCHIES OF COUNTABLE PRODUCTS OF POLISH SPACES

1. Introduction. Let X be an uncountable Polish (complete, separable metric) space and let $H = X^{\omega}$. Equip H with the product of discrete topologies and also with the product of copies of the Polish topology. The former topology will be called the d-topology and the latter the p-topology (which is known to be Polish). Topological properties with respect to these topologies will carry the prefix d and p as the case may be. The d- topology on H gives rise to two hierarchies (of Borel sets) defined as follows.

Put $$\Sigma_0 = \Pi_0 = \left\{ A \subseteq H : A \text{ is } d\text{-clopen} \right\},$$

and inductively define for $\mu < \omega_1$,

$$\Sigma_\mu = \Big(\bigcup_{\nu<\mu} \Pi_\nu \Big)_\sigma;$$
$$\Pi_\mu = \{ A : \ A^c \in \Sigma_\mu \}.$$

Denoting the Borel σ—field on H with respect to p by $\mathfrak{B}$, we define

$$\Sigma_0^* = \Pi_0^* = \left\{ A \in \mathfrak{B} : A \text{ is } d\text{-clopen} \right\};$$
$$\Sigma_\nu^* = \Big(\bigcup_{\nu<\mu} \Pi_\nu^* \Big)_\sigma;$$
$$\Pi_\mu^* = \{ A : \ A^c \in \Sigma_\mu^* \}.$$

It is not hard to check (as pointed out to us by a referee) that

$$\mathfrak{B} = \bigcup_{\mu<\omega_1} \Sigma_\mu^* .$$

A.Maitra [2] asked whether

(I) $$\Sigma_\mu^* = \Sigma_\mu \cap \mathfrak{B} \qquad \text{for } 0 \le \mu < \omega_1.$$

In connection with (I) he made the following conjecture (which he proved for the case $\mu = 1,2$).

(II) Suppose A and B are two analytic (Σ_1^1) subsets of H such that A can be separated from B by a Σ_μ set, $0 \leq \mu < \omega_1$. Then there is a Σ_μ^* set which separates A from B.

Observe that trivially (II) implies (I). In this short note we shall show that under a certain category- theoretic assumption, (II) is true for all $\mu < \omega_1$. We shall obtain this as a simple consequence of results and tecniques of Louveau developed in [1]. The relevent definitions and results are reviewed in the next section.

Note that the above assumption is not consistent with ZFC. However, it holds in the Lévy- Solovay model [4].

2. Regular and Separating Families of Sets. We shall use standard notation and terminology from effective descriptive set theory as found in Moschovakis [3]. All unexplained notation and terminology are from [3].

Definition 1. Let X be a recursively presentable (r.p.) space. By a coding pair we shall mean a pair $< W^X, C^X >$ (or $< W, C >$ when X is clear from the context) such that

(i) W is a Π_1^1 subset of $\omega^\omega \times \omega$.

(ii) C is a Π_1^1 subset of $X \times \omega^\omega \times \omega$ whose projection on $\omega^\omega \times \omega$ is W and such that the relation

$$(\alpha, n) \in W \ \& \ (x, \alpha, n) \notin C$$

is Π_1^1.

(*iii*) For each α, $\{ C_{\alpha,n} : n \in \omega, (\alpha, n) \in W \}$ is precisely the class of all $\Delta_1^1(\alpha)$ subsets of X. Observe that such a coding pair exists (*cf*.[1; p14]).

Definition 2 (*Louveau*). A family Φ of subsets of an r.p. space X is said to be *separating* with parameter $\alpha_0 \in \omega^\omega$, if it satisfies the following two conditions:

(i) The set $W_\Phi \overset{def}{=} \{ (\alpha,n) \in W : C_{\alpha,n} \in \Phi \}$ is a $\Pi^1_1(\alpha_0)$ set .

(ii) If A_1 and A_2 are two $\Sigma^1_1(\alpha)$ subsets of X and if there is a $B \in \Phi$ which separates A_1 from A_2, then there exists a $\Delta^1_1(<\alpha_0,\alpha>)$ set in Φ which separates A_1 from A_2.

Definition 3 (*Louveau*). Let Φ be a family of subsets of a r.p. space X and $\alpha \in \omega^\omega$. The *separating kernel of* Φ *of order* α, written $S_\alpha(\Phi)$, is the family of $\Sigma^1_1(\alpha)$ subsets of X which can be separated from *every* disjoint $\Sigma^1_1(\alpha)$ set by a set which is $\Delta^1_1(\alpha)$ and in Φ

Clearly, $\Delta^1_1(\alpha) \cap \Phi \subseteq S_\alpha(\Phi)$; and if Φ is separating, then $\Sigma^1_1(\alpha) \cap \Phi \subseteq S_\alpha(\Phi)$.

Notation. For each r.p. space X , let $T^X(\alpha)$ denote the topology generated by the $\Sigma^1_1(\alpha)$ subsets of X. We shall often drop the superscript when it is clear from the context. *Note that if* X *is a product space the* $T^X(\alpha)$ *is not the product topology. In what follows , unless explicitly mentioned, we shall always use this topology and not the product topology when* X *is a product space.* For $A, B \subseteq X$, we write $A \sim_\alpha B$ iff $A \triangle B$ ($= (A-B) \cup (B-A)$) is $T(\alpha)$ - meager.

Definition 4 (*Louveau*). Let Φ be a family of subsets of an r.p. space X. Then Φ is said to be *regular* with parameter $\alpha_0 \in \omega^\omega$, if it satisfies the following two properties:

(i) The set W_Φ (*cf.* Definition 2) is $\Pi^1_1(\alpha_0)$.

(*ii*) *Property of regularity*: For every real α and for every set $E \in \Phi$, there is a sequence $\{ A_n : n \in \omega \} \subseteq S_{<\alpha_0,\alpha>}(\Phi)$ such that

$$E \sim_{<\alpha_0,\alpha>} (\cup_n A_n).$$

Theorem 1 (*Louveau*). For each $\mu < \omega_1$, define a family Φ_μ by the recursion:

$$\Phi_0 = \Phi ;$$
$$\Phi_{\mu+1} = (\Phi_\mu)_{\sigma c} ;$$
$$\Phi_\lambda = \bigcup_{\mu<\lambda} \Phi_\mu , \text{ if } \lambda \text{ is limit.}$$

Then, each Φ_μ is regular (and for $\mu > 0$, separating) if Φ is regular, with parameter $< \alpha_0, \alpha_\mu >$, where α_0 is the parameter of Φ and $\alpha_\mu \in WO$ is a real such that $|\alpha_\mu| = \mu$.

Moreover, let $\phi(\alpha, n, E)$ be the following set relation:

$$\phi(\alpha, n, E) \longleftrightarrow$$
$$\left[(\alpha,n)\in W\right]\&\left[\exists\beta\in\Delta^1_1(<\alpha_0,\alpha>)(\forall m)\left[(<\alpha_0,\alpha>,\beta(m))\in E \;\&\; C_{\alpha,n} = X - \bigcup_m C_{<\alpha_0,\alpha>,\beta(m)}\right]\right]$$

Plainly, ϕ is $\Pi^1_1(\alpha_0)$—monotone which defines inductively a sequence W^μ_Φ by

$$W^0_\Phi = W_\Phi ;$$
$$W^{\mu+1}_\Phi = \left\{(\alpha, n) : \phi(\alpha, n, W^\mu_\Phi)\right\};$$
$$W^\lambda_\Phi = \bigcup_{\mu<\lambda} W^\mu_\Phi , \text{ if } \lambda \text{ is limit.}$$

Then, for each ordinal μ,

$$W_{\Phi_\mu} = W^\mu_\Phi .$$

(For a proof of this theorem see [1]).

Proof of Conjecture (II) . *Without loss of generality, we assume that* $X = \omega^\omega$; the result for Polish spaces can be obtained by standard transfer theorems. We shall identify $X^n \times X^\omega$ with X^ω for each integer $n \geq 1$.

We first make the following easy but important observation.

Proposition. Any set $A \in \Sigma_1$ is of the form $A = \bigcup_{n\geq 1} A_n$, with each $A_n = A'_{(n)} \times X^{\omega}$ with $A'_{(n)} \subseteq X^n$.

Thus Σ_1 precisely consists of sets of the above type.

Proof. Let $A'_{(n)} = \{ (x_1, \cdots, x_n) : \Sigma(x_1, \cdots, x_n) \subseteq A \}$, where $\Sigma(\mathrm{x}_1, \cdots, \mathrm{x}_n) = \left\{ \mathbf{x} \in X^{\omega} : (\mathbf{x})_1 = \mathrm{x}_1, \ldots, (\mathbf{x})_n = \mathrm{x}_n \right\}$. Clearly, $A'_{(n)} \times X^{\omega} \subseteq A$ for every n. Now suppose $\mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots) \in A$. Since A is d-open, there is $n \geq 1$ such that

$$\mathbf{x} \in \Sigma(\mathrm{x}_1, \cdots, \mathrm{x}_n) \subseteq A.$$

Clearly $(x_1, \cdots, x_n) \in A'_{(n)}$ and hence $\mathbf{x} \in A'_{(n)} \times X^{\omega}$. □

Let $\Phi_0 = \{ A'_{(n)} \times X^{\omega} : n > 0 \ \& \ A'_{(n)} \subseteq X^n \}$ and inductively define Φ_μ as in **Theorem 1**. It is not hard to see that

$$\Pi_\mu = \begin{cases} \Phi_\mu & \text{if } 1 \leq \mu < \omega_0 ; \\ \Phi_{\mu+1} & \text{if } \mu \geq \omega_0 . \end{cases}$$

Now consider the following statement:

($\mathfrak{P}$) For each $n \geq 1$, every subset A of $(\omega^{\omega})^n$ has the Baire property relative to $T(\alpha)$ for every α.

Observe that under the Axiom of Choice statement ($\mathfrak{P}$) is not true . However, in the Lévy-Solovay model [4] , it can be shown that ($\mathfrak{P}$) holds (*cf.*[1]). As pointed out to us by a referee AD implies ($\mathfrak{P}$)— this can be proved by playing the Banach - Mazur game with $\Sigma^1_1(\alpha)$ sets.

We now prove

Lemma. Assume that ($\mathfrak{P}$) holds. Then the family Φ_0 is regular without parameter.

Proof. Let $< W^m, C^m >$ be (uniformly in m) a coding pair for X^m and let $<W, C>$ be a coding pair for X^ω. Observe that $A'_{(n)} \subseteq X^n$ is $\Delta^1_1(\alpha)$ in X^n iff $A'_{(n)} \times X^\omega$ is $\Delta^1_1(\alpha)$ in X^ω. Hence

$$(\alpha, n) \in W_{\Phi_0}$$

$$\longleftrightarrow \quad (\alpha, n) \in W \;\&\; C_{(\alpha, n)} \in \Phi_0$$

$$\longleftrightarrow \quad (\alpha, n) \in W \;\&\; (\exists m)(\exists k)\{ (\alpha, k) \in W^m \;\&\; C^m_{\alpha, k} \times X^\omega = C_{\alpha, n} \}.$$

It is easy to check that W_{Φ_0} is Π^1_1. Next observe that if $E \subseteq X^n$ is $\Sigma^1_1(\alpha)$, then $E \times X^\omega$ is in $S_\alpha (\Phi_0)$. This follows from the Suslin-Kleene Theorem(*cf.* [3]).

Now, fix α and let $A \in \Phi_0$. Then for some n, $A = A'_{(n)} \times X^\omega$, where $A'_{(n)} \subseteq X^n$. Since ($\mathfrak{P}$) holds, there is a sequence $\{ E_k \}$ of $\Sigma^1_1(\alpha)$ subsets of X^n such that $A'_{(n)} \sim_\alpha (\bigcup_k E_k)$. Hence,

$$(A'_{(n)} \times X^\omega) \,\Delta\, (\bigcup_k (E_k \times X^\omega)) = (A'_{(n)} \,\Delta\, (\bigcup_k E_k)) \times X^\omega$$

is meager relative to $T(\alpha)$ by *Lemma* 2.13 of [1]. But each $E_k \times X^\omega \in S_\alpha (\Phi_0)$ by the observation above. Thus A has the regularity property.

This completes the proof.

Theorem 2. Assume ($\mathfrak{P}$). Suppose A_1 and A_2 are two Σ^1_1 subsets of H such that A_1 can be separated from A_2 by a Π_μ set with $\mu \geq 1$. Then A_1 can be separated from A_2 by a Π^*_μ set.

In other words, Conjecture (II) is true for all $\mu < \omega_1$.

Proof. We shall prove this by induction on μ. The result for $\mu = 1$ is known and can be easily proved. So assume $\mu > 1$ and fix z such that $\mu < \omega_1^z$ and A_1, A_2 are $\Sigma^1_1(z)$ subsets of H.

Now observe that, by the **Lemma** and **Theorem 1**, Π_μ is a separating family with parameter α_μ (which can be chosen to be recursive in z). Hence there is a set B separating A_1 from A_2 such that B is $\Delta_1^1(<\alpha_\mu, z>)$ and in Π_μ. Fix n such that $(<\alpha_\mu, z>, n) \in W^H$ and $C^H_{<\alpha_\mu, z>, n} = B$. Plainly, $(<\alpha_\mu, z>, n) \in W_{\Pi_\mu} = W^{\mu+1}_{\Phi_0}$, by Theorem 1. (We assume for simplicity that $\mu \geq \omega$).

Hence there exists $\beta \in \Delta_1^1(<\alpha_\mu, z>)$ such that

$$(\forall m)\left[(<\alpha_0,\alpha>,\beta(m)) \in W^{\mu}_{\Phi_0}\right] \;\&\; C^H_{<\alpha_\mu, z>, n} = H - \bigcup_m C^H_{<\alpha_\mu, z>, \beta(m)} .$$

Write $B_m = C^H_{<\alpha_\mu, z>, \beta(m)}$ for each m. Clearly, each B_m is $\Delta_1^1(<\alpha_\mu, z>)$ and in Π_{η_m} for some $\eta_m < \mu$. By induction hypothesis, $B_m \in \Pi^*_{\eta_m}$ and hence $\bigcup_m B_m \in \left(\bigcup_{\eta<\mu} \Pi^*_\eta\right)_\sigma = \Sigma^*_\mu$. Thus $B = H - \cup B_m$ is a Π^*_μ set which separates A_1 from A_2. This completes the proof.

As an immediate consequence we have

Corollary. Assume ($\mathfrak{P}$). Then, for $\mu < \omega_1$,

$$\Sigma^*_\mu = \Sigma_\mu \cap \mathfrak{B} .$$

Remark. As remarked earlier, the statement ($\mathfrak{P}$) holds in the Lévy-Solovay model. Consequently, both Conjectures (I) and (II) are true in that model.

Postscript. V.V. Srivatsa (unpublshed) has proved (I) in ZFC.

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Received March 30, 1989