Parametric $\mathcal{I}$-approximate derivatives are in Baire class one

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It is well-known that if a function $f : \mathbb{R} \to \mathbb{R}$ is approximately differentiable by any of the common methods, then its approximate derivative is in Baire class one. In fact, this observation has been proved to be true for a large class of generalized derivatives which are definable via difference quotients by the following method [1].

Definition. *A function h is a parameter function iff it satisfies:*
(1) h is measurable;
(2) if G is any measurable set, then $h(G)$ is measurable; and,
(3) if A_n is any sequence of Borel sets such that $\lim_{n\to\infty} |A_n| = 0$, then

$$\lim_{n\to\infty} |h^{-1}(A_n)| = 0.$$

Definition. *A generalized difference quotient is a quotient of the form*

$$Q(x,t) = \frac{1}{g(t)} \sum_{i=1}^{n} a_i f(x + h_i(t)),$$

where $a_i \in \mathbb{R}$, h_i are parameter functions, $1 \le i \le n$, and f, g are arbitrary functions.

Definition. *The generalized derivative associated with this general difference quotient is*

$$f^*(x) = \operatorname*{lap}_{t\to 0} Q(x,t)$$

whenever this limit exists, where "lap" stands for the approximate limit.

Using this structure, the following theorem can be proved.

Theorem 1. *If $f : \mathbb{R} \to \mathbb{R}$ is a measurable functions such that f^* exists everywhere (finite or infinite), then f^* is in Baire class one.*

In a simple case the proof of this theorem can be hinted at by the following example.

Example. *Let*

$$Q(x,h) = \frac{f(x+h) - f(x-h)}{2h}.$$

The idea of the proof in this case is a "balancing act." The set

$$S_{rst} = \left\{ x : \frac{|\{h \in (0,1/r) : Q(x,h) > -1/s\}|}{1/r} > 1 - \frac{1}{t} \right\}, \quad r,s,t \ge 1,$$

is open, and

$$\{x : f(x) \ge 0\} = \bigcap_{t=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} S_{rst} \in \mathbf{G}_\delta.$$

But, Theorem 1 can also be used in cases where the "quotient" structure is less obvious.

Example. *$Q(x,h) = f(x+h)$ yields that approximately continuous functions are in Baire class one.*

A natural question is whether the same methods can be used to prove that similar results hold for the $\mathcal{I}$-approximate limit [2] instead of the approximate limit. The definitions given above can be paralleled in the following manner.

Definition. *A function h is an $\mathcal{I}$-parameter function iff:*
(1) h is $\mathcal{I}$-approximately continuous;
(2) if G has the Baire property, then $h(G)$ has the Baire property; and,
(3) if A_n is any sequence of open sets such that $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{I}$, then

$$\bigcap_{n\in\mathbb{N}} h^{-1}(A_n) \in \mathcal{I}.$$

Define $Q(x,t)$ as before and $f^*_{\mathcal{I}}$ in the natural way. Using these definitions, the following theorem analogous to Theorem 1 is true.

Theorem 2. *If f is a function with the Baire property such that $f^*_{\mathcal{I}}$ exists everywhere (finite or infinite), then $f^*_{\mathcal{I}}$ is in Baire class one.*

Corollary. *In particular, the following types of "derivatives" are all in Baire class one.*
(1) $\mathcal{I}$-approximately continuous functions.
(2) The n^{th} $\mathcal{I}$-approximate symmetric derivatives (with infinite values allowed), when n is odd.
(3) The n^{th} finitely $\mathcal{I}$-approximately symmetrically differentiable functions, where n is even.

Finally, the following can be said about the extreme $\mathcal{I}$-approximate derivatives.

Theorem 3. *If $\overline{f}^*_{\mathcal{I}}$ and $\underline{f}^*_{\mathcal{I}}$ are the upper and lower $\mathcal{I}$-approximate derivatives of a function f, with the Baire property, then $\overline{f}^*_{\mathcal{I}}$ and $\underline{f}^*_{\mathcal{I}}$ are both in Baire class three.*

References

[1] Lee Larson, A method for showing generalized derivatives are in Baire class one. *Contemporary Math.*, 42:1985.

[2] W. Wilszynski, A category analogue of the density topology, approximate continuity, and the approximate derivative. *Real Anal. Exch.*, 10:241–265, 1984–85.