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EXTREME POINT SELECTORS

In [1], L. Baggett states and proves the following selection lemma.

Lemma. [1] Let X be a separable normed linear space, let Y be a closed subspace of X and let R denote the restriction map of X^* onto Y^*. Let K be a compact subset of $(X^*,w*)$, and let $L=R(K)$. Then there exists a Borel map $s:L\to K$ such that $R(s(y)) = y$ for all $y\in L$, $s(y)$ is an extreme point of $R^{-1}(y)$ and $y\in ext(L) \to s(y)\in ext(K)$.

The proof of this theorem will follow from our main result.

Theorem. Let X be a separable normed linear space with continuous dual X^*. Let $\mathscr{C}(X^*)$ denote the collection of all nonempty, compact, convex subsets of $(X^*,w*)$ where $w*$ denotes the weak–star topology. We give $\mathscr{C}(X^*)$ the relative exponential (Vietoris) topology. Define the multifunction $E: \mathscr{C}(X^*) \to X^*$ by $E(K) = ext\ K \subset K$, where ext K denotes the set of extreme points of K. Then there is a Borel class 1 selector for E.

Let B denote the unit ball in X^*, i.e., $B = \{ x\in X^* : \|x\|\leq 1 \}$. By Alaoglu's theorem (see [3,p.202]), B is $w*$–compact. Furthermore, since X is separable, B is metrizable. Set $B_n = n\cdot B$ for each $n\in N$. Then for each n, B_n is compact and metrizable. Let $\mathscr{C}(B_n)$ denote the collection of all nonempty, convex, compact subsets of B_n. We give $\mathscr{C}(B_n)$ the relative exponential (Vietoris) topology. Define for each n, $E_n : \mathscr{C}(B_n) \to B_n$ by $E_n(K) = ext\ K$. Note that by the Krein–Milman theorem (see [3,p.207]), $\forall K\in \mathscr{C}(B_n)$, $E_n(K)\neq\phi$ and $K = cl\ conv(ext\ K)$. We claim that for each n, E_n has a Borel class 1 selector. This will follow from a theorem of G. Debs.

Theorem [2]. Suppose T is a metric space, X Polish, Gr(F) is a G_δ, $\alpha>0$ is an ordinal, and $F^{-1}(U)$ is of additive class α for open $U \subset X$. Then F has a selection which is of additive class α.

Let us note that $\mathscr{K}(B_n)$ is Polish, and $Gr(E_n)$ is a G_δ subset of $\mathscr{K}(B) \times B$. An additional fact is that E_n is lower semi–continuous, i.e., $E_n^{-1}(V)=\{K \in \mathscr{K}(X^*):E_n(K) \cap V \neq \phi\}$ is open whenever V is open. Therefore, by Deb's theorem, there is for each n a Borel class 1 selector f_n for E_n. Define $f: \mathscr{K}(X^*) \to X^*$ as follows:

$$f(K) = f_n(K) \text{ where } n=\text{least}\{m:K \subset B_m\}.$$

Then $\forall K \in \mathscr{K}(X^*)$, $f(K) \in E(K)$. Furthermore, f is of Borel class 1, since

$$f^{-1}(A) = f_1^{-1}(A) \cup [\bigcup_{i \geq 2} f_i^{-1}(A) \cap (\mathscr{K}(B_{i-1}))^c]$$

for any $A \subset X^*$ and since each f_n is of Borel class 1. Hence, f is a Borel class 1 selector for E.

Lastly, we mention that the above result is sharp. In fact, E does not have a continuous selector in the case $X=\mathbb{R}^2$.

REFERENCES

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