REFINEMENTS OF THE DENSITY AND I-DENSITY TOPOLOGIES.

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Definitions.

(1) A point $x \in \mathbf{R}$ is a density point of $A \subset \mathbf{R}$ if

$$\lim_{h\to 0^+} \frac{m_i((x-h,x+h)\cap A)}{2h} = \lim_{n\to\infty} \frac{m_i((x-\frac{1}{n},x+\frac{1}{n})\cap A)}{2/n} = 1,$$

where m_i stands for inner Lebesgue measure.

(2) $\Phi(A) = \{x \in \mathbf{R}$: is a density point of $A\}$.

(3) $T_N = \{A \subset \mathbf{R}:\ A \subset \Phi_N(A)\} \subset L$ is a density topology, where L is a family of all Lebesgue measurable sets.

Alternative definitions of the density points.

For a measurable set $A \subset \mathbf{R}$ the following are equivalent:

(A) 0 is a density point of A;

(B) $\lim_{n\to\infty} \frac{m((x-\frac{1}{n},x+\frac{1}{n})\cap A)}{2/n} = 1$;

(C) $\lim_{n\to\infty} m(nA \cap (-1,1)) = 2$;

(D) $\chi_{nA \cap (-1,1)}$ converges to $\chi_{(-1,1)}$ in measure;

(E) $\forall\ \{n_m\} \subset N,\ n_m \to \infty\ \exists\ \{n_{m_p}\}\ \lim_{p\to\infty} \chi_{n_{m_p}A\cap(-1,1)} = \chi_{(-1,1)}$ almost everywhere;

and

(B') $\lim_{n\to\infty} \frac{m((x-1/t_n,x+1/t_n)\cap A)}{2/t_n} = 1$ for all $t_n>0$, $t_n \to \infty$;

(C') $\lim_{n\to\infty} m(t_nA \cap (-1,1)) = 2$; for all $t_n>0$, $t_n \to\infty$;

(D') $\chi_{t_n A \cap (-1,1)}$ converges to $\chi_{(-1,1)}$ in measure for all $t_n>0$, $t_n\to\infty$;

(E') $\forall\ \{t_n\} \subset \mathbf{R}^+,\ t_n \to \infty\ \exists\ \{t_{n_p}\}\ \lim_{p\to\infty} \chi_{t_{n_p}A\cap(-1,1)} = \chi_{(-1,1)}$ almost everywhere.

Definitions. Let J be an ideal of subsets of reals.

(1) A sequence $f_n: \mathbf{R}\to\mathbf{R}$ converges (J) to f if for every subsequence f_{n_p} of f_n there exists a further subsequence $f_{n_{p_q}}$ such that $f_{n_{p_q}}$ converges pointwise to f J-a.e..

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(2) A point a is a <u>*J*-density point</u> of a set $A \subset \mathbf{R}$ if

$$\chi_{n(A-a) \cap (-1,1)} \text{ converges } (J) \text{ to } \chi_{(-1,1)}.$$

$\Phi_J(A) = \{x \in \mathbf{R}: \text{ is a } J\text{-density point of } A\}$.

$T'_J = \{A \subset \mathbf{R}: A \subset \Phi_J(A)\}$.

(3) A point a is a <u>strong *J*-density point</u> of a set $A \subset \mathbf{R}$ if for all $t_n > 0$, $t_n \to \infty$,

$$\chi_{t_n(A-a) \cap (-1,1)} \text{ converges } (J) \text{ to } \chi_{(-1,1)}.$$

$\Psi_J(A) = \{x \in \mathbf{R}: \text{ is a strong } J\text{-density point of } A\}$.

$T''_J = \{A \subset \mathbf{R}: A \subset \Psi_J(A)\}$.

(4) $T_I = T'_I \cap B = T''_I \cap B$ is an *I*-density topology, where *I* is the ideal of first category sets and *B* is the family of all sets with Baire property.

Evidently,

$$T''_J \subset T_J \quad \text{for every } J,$$

$$T_N \subset T'_N \subset T_N \quad \text{and} \quad T_I \subset T'_I \subset T_I.$$

<u>THEOREM</u>. Families T'_J and T_J form topologies on $\mathbf{R}$.

<u>Example 1</u>. There exists a nonmesurable set $A \subset \mathbf{R}$, which does not have the Baire property such that

$$\lim_{n\to\infty} \chi_{n(A-a) \cap (-1,1)} = \chi_{(-1,1)} \quad \text{for every } a \in A.$$

In particular, $A \in T'_J$ for every *J*.

<u>Corollary 1</u>. $T'_N \not\subset T_N$ and $T'_I \not\subset T_I$.

<u>Example 2</u>. There exists a nonmesurable set $A \subset \mathbf{R}$, which does not have the Baire property such that $A \subset \Psi_{I_c}(A)$, where I_c is the ideal of sets of cardinality less then continuum c.

<u>Corollary 2</u>. If the Continuum Hypothesis, or Martin's Axiom, holds then

$$T''_N \not\subset T_N \text{ and } T'_I \not\subset T_I.$$

The above examples and an obvious equation

$$T_N = T'_N \cap L = T''_N \cap L$$

justify the definition (4) of an *I*-density topology as

$$T_I = T_I \cap B = T''_I \cap B.$$

Should we define, for an arbitrary ideal *J*, a *J*-density topology as

$$T_J = T''_J \cap F_J$$

for some family F_J? If so, the family F_J should satisfy the condition that

(*) $\quad T''_J \cap F_J$ forms a topology on $\mathbf{R}$.

It would be also very desired to have a condition

(**) $\quad T''_J \cap F_J = T'_J \cap F_J$.

<u>Example 3</u>. $T_{\{\emptyset\}} \cap F_\sigma \cap G_\delta \not\subset T''_{\{\emptyset\}}$ = natural topology.

<u>Example 4</u>. Let I_ω stand for the ideal of countable sets. Then topology generated by

$T''_{I_\omega} \cap F_\sigma \cap G_\delta$ contains non-Borel set. Moreover, $T_{I_\omega} \cap F_\sigma \cap G_\delta \not\subset T''_{I_\omega}$.