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Some results and problems about ω-limit sets

In [ABCP] it was proven that a nonvoid closed subset F of $I = [0,1]$ is an ω-limit set $\omega(x_0, f)$ for some continuous $f : I \to I$ iff F is nowhere dense or a union of finitely many nondegenerate closed intervals. In the proof it turned out that the limit points of F were fixed points of f in the case the isolated points of F were dense in F. What, then, is a characterization of the set of fixed points for a continuous function f realizing a given F as one of its ω-limit sets? If F consists of more than one interval f clearly has no fixed points in F. The following results handle the remaining cases.

THEOREM 1 [AC3]. *The set K is the set of fixed points for some f realizing I as an ω-limit set iff K is a nonvoid nowhere dense closed subset of I different from $\{0,1\}, \{0\}$ and $\{1\}$.*

THEOREM 2 [C]. *Suppose M is a nonvoid closed nowhere dense subset of I and $K \subseteq M$. Then, M is an ω-limit set for some f having K as its set of fixed points in M iff K is closed and nowhere dense in M and only one of the following hold*

(1) *$K = \phi$ and there is more than one point of highest order in M*
(2) *$K \neq \phi$, $M - K$ is countable and it is not the case that ρ has an absolute maximum on $M - K$ occuring only at one point*
(3) *$K \neq \phi$ and $M - K$ is uncountable*

The order $\rho(x)$ at a point x is the smallest α such that $x \in D_\alpha(M)$. For $\alpha \leq \omega$, we define $D_\alpha(A)$ as follows: $D_0(A) = A$, $D_{\alpha+1} = D(D_\alpha(A))$ and $D_\lambda(A) = \bigcap\{D_\alpha(A) : \alpha < \lambda\}$ when λ is a limit ordinal where $D(B)$ is the set of limit points of B.

In [AC1] the study of compact ω-limit sets in E^k was initiated. One unresolved problem is the analogous one of how to characterize the fixed points of continuous functions realizing a given compact planar set as one of its ω-limit sets. Another more basic problem is characterizing those compact sets in E^k which can be ω-limit sets for continuous functions from E^k into E^k.

We know (1) [AC1] *each nonvoid continuum in E^k with empty interior is an ω-limit set* and (2) [AC2] *each nonvoid locally connected continuum in E^k is an ω-limit set.* These results and several examples suggest the following

<u>*Conjecture*</u> *1 A nonvoid continuum in E^k is an ω-limit set iff it has void interior or is locally (or arcwise) connected.*

<u>*Conjecture*</u> *2 A nonvoid compact set M in E^k with finitely many components $M_1, \ldots, M_n$ is an ω-limit set iff M_1 is an ω-limit set and there exists a continuous $f : M \to M$ such that $f(M_i) = M_{i+1} \pmod{n}$.*

As far as compact sets with infinitely many components goes we know that *any ω-limit set with infinitely many components has empty interior* and [AC1] *any totally disconnected compact set in E^k is an ω-limit set.*

The following conjecture is true in E^1 [BS] and seems reasonable in general

<u>Conjecture</u> *3 A compact set M in E^k with infinitely many components is an ω-limit set iff there exists finitely many nowhere dense components $K_1, \ldots, K_n$ and a sequence of mutually disjoint nowhere dense compact subsets $\{C_i\}_{i=1}^{\infty}$ of M such that $\bigcup_{i=1}^{\infty} C_i = M - \bigcup_{i=1}^{n} K_i$ and there exists a continuous $f : M \to M$ such that $f(C_{i+1}) = C_i$ and $f(K_i) = K_{i+1} \pmod{n}$ for each i.*

We say that $\omega(x_0, f)$ is <u>orbit-enclosing</u> if $\omega(x_0, f)$ contains a tail of $\{f^n(x_0)\}_{n=0}^{\infty}$. If $\omega(x_0, f)$ has nonvoid interior it must be orbit enclosing. Another unsolved problem is how to characterize the orbit enclosing compact ω-limit sets. In this respect we offer

<u>Conjecture</u> *4 A nonvoid continuum in E^k is an orbit enclosing ω-limit set iff it is arcwise connected.*

References

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