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Three Methods of Constructing ω-limit Sets*

Let f be a function mapping $\mathbf{I} = [0, 1]$ into itself. A set $\mathbf{K} \subset \mathbf{I}$ is called an ω-limit set for f if there exists $x \in \mathbf{I}$ such that $\mathbf{K}$ is the cluster set of the sequence $\{f^n(x)\}$. (Here, as usual, $f^1 = f$ and $f^{n+1} = f \circ f^n$ for $n = 1, 2, 3, \ldots$). We write $\omega_f(x) = \mathbf{K}$ to indicate $\mathbf{K}$ is the ω-limit set of x under f.

Let $\mathcal{K}$ be a nonempty family of compact subsets of $\mathbf{I} = [0, 1]$, and let $\mathfrak{F}$ be a family of functions from $\mathbf{I}$ to $\mathbf{I}$. For $f \in \mathfrak{F}$, let $\Lambda(f)$ denote the class of ω-limit sets for f. We discuss variants of the following question. Given $\mathcal{K}$ and $\mathfrak{F}$, does there exist $f \in \mathfrak{F}$ such that $\mathcal{K} \subset \Lambda(f)$? We pay particular attention to subfamilies $\mathfrak{F}$ of $\mathcal{DB}_1$.

We consider three methods of analysis:

1. Arithmetic.

Here one constructs functions whose iterative patterns are built into the, say, ternary representations for numbers in $\mathbf{I}$. This method is useful for constructing examples that illustrate that large families $\mathcal{K}$ can be incorporated into $\Lambda(f)$ for some $f \in \mathfrak{F}$

*A more detailed summary of this talk is included in the Inroads section of this issue.

2. Interval-orbits.

Here one constructs a nested sequence $\{I_n\}$ of compact intervals. The orbit of I_n approximates the desired ω-limit set $\mathbf{K}$ for a finite number i_n of iterates. As n increases, the approximation improves in two ways: the error tolerances decrease to 0, and the numbers i_n approach infinity. The point $\{x\} = \bigcap I_n$ has $\mathbf{K}$ as ω-limit set. Using this approach one can show that there exists a function f having only one point of discontinuity and possessing the Darboux property such that $\Lambda(f)$ contains a homeomorphic copy of each nonempty nowhere dense compact set as well as copies of those sets that are finite unions of closed intervals.

3. Specifying Orbits.

We determine conditions on a sequence $S = \{x_n\}$ or on a countable collection $\mathcal{A}$ of sequences that allow each $S \in \mathcal{A}$ to be the orbit of some $x(S) \in I$ for some $f \in \mathfrak{F}$. Given a countable family $\mathfrak{K} = \{K_1, K_2, \ldots\}$ one tries to choose a countable collection $\mathcal{A}$ of sequences$\{S_1, S_2, \ldots\}$ such that the sequence S_n has the set K_n as cluster set and such that the collection $\mathcal{A}$ allows the existence of a function $f \in \mathfrak{F}$ that realizes each sequence S_n as the orbit of some point. This method has been useful, for example, in showing that a given nonempty nowhere dense set K is an ω-limit set for some $f \in \mathfrak{C}$. For $\mathfrak{F} = \mathfrak{DB}_1$, this method readily reveals that any nonempty compact K is an ω-limit set for some $f \in \mathfrak{DB}_1$. It is also helpful in identifying conditions under which a countable family $\mathfrak{K}$ is contained in $\Lambda(f)$ for some $f \in \mathfrak{DB}_1$.