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AN INTEGRAL IN GEOMETRIC MEASURE THEORY

Throughout, $m \geq 1$ is a fixed integer. The set of all real numbers is denoted by $\mathbf{R}$, and the m-fold Cartesian product of $\mathbf{R}$ is denoted by $\mathbf{R}^m$. In $\mathbf{R}^m$ we shall consider the m-dimensional Lebesgue measure λ and the $(m-1)$-dimensional Hausdorff measure $\mathcal{H}$. The use of the Lebesgue integral (with respect to λ or $\mathcal{H}$) is indicated by the symbol $(L)\int$; the symbol $\int$ is reserved for the new integral defined below.

The *essential closure* of a set $E \subset \mathbf{R}^m$, denoted by E^*, is the set of all $x \in \mathbf{R}^m$ at which the upper density of E with respect to λ is positive; the *essential boundary* of E is the set $\partial^* E = E^* \cap (\mathbf{R}^m - E)^*$. A BV set is a bounded subset of $\mathbf{R}^m$ for which $\mathcal{H}(\partial^* A) < +\infty$. According to [**F**, Theorem 4.5.11], the family of all BV sets coincides with the family of all bounded λ-measurable subsets of $\mathbf{R}^m$ whose De Giorgi *perimeters* defined in [**M-M**, Chapter 2] are finite; moreover, the perimeter of a BV set A equals to $\mathcal{H}(\partial^* A)$ (see [**V**, Section 4]). Denoting by $d(A)$ and $\|A\|$, respectively, the diameter and perimeter of a BV set A, we define the *regularity* of A as the number

$$r(A) = \begin{cases} \frac{\lambda(A)}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using the isoperimetric inequality, it is easy to relate $r(A)$ to the usual concept of regularity connected with Vitali's covering theorem (cf. [**S**, Chapter IV, Section 2]).

A *partition* in a BV set A is a collection (possibly empty) $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint BV subsets of A and $x_i \in A_i$ for $i = 1, \ldots, p$. When $r(A_i) > \varepsilon$ for an $\varepsilon > 0$ and $i = 1, \ldots, p$, we say that the partition P is *ε-regular.* If δ is a nonnegative function on A^* and $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$, then P is called *δ-fine.* Finally, given an $\varepsilon > 0$ and a sequence $\eta = \{\eta_j\}$ of positive numbers, we say that P is *(ε, η)-approximating* whenever there are disjoint BV sets $B_1, \ldots, B_k$ such that $A - \bigcup_{i=1}^p A_i = \bigcup_{j=1}^k B_j$ and for $j = 1, \ldots, k$, we have $\|B_j\| < 1/\varepsilon$ and $\lambda(B_j) < \eta_j$.

NOTE. If a partition $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ in a BV set A is δ-fine, a two-fold limitation is implied: the points $x_1, \ldots, x_p$ lie outside the set $\{x \in A^* : \delta(x) = 0\}$ and the diameters of the sets A_i are bounded by $\delta(x_i)$ for $i = 1, \ldots, p$. We shall use "small" functions δ which vanish on "small" sets in terms of the measure $\mathcal{H}$. In general, $\bigcup_{i=1}^p A_i$ is a proper subset of A. If P is (ε, η)-approximating, however, then $\bigcup_{i=1}^p A_i$ fills "most" of A with respect to the measure λ, as we shall employ sequences η of "small" numbers.

A *gage* in a BV set A is a nonnegative function δ on A^* such that the set $\{x \in A^* : \delta(x) = 0\}$ is a countable union of sets whose $\mathcal{H}$ measure is finite. For any gage δ in a BV set A, any sequence η of positive numbers, and any sufficiently small $\varepsilon > 0$, the existence of δ-fine ε-regular and (ε, η)-approximating partitions in A was established in [$\mathbf{P}_2$].

DEFINITION. Let A be a BV set. A function $f : A^* \to \mathbf{R}$ is called *integrable* in A if there is a number α satisfying the following condition: given $\varepsilon > 0$, there is a sequence η of positive numbers and a gage δ in A such that

$$\left| \sum_{i=1}^{p} f(x_i)\lambda(A_i) - \alpha \right| < \varepsilon$$

for each partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A which is simultaneously ε-regular, δ-fine, and (ε, η)-approximating. The number α, necessarily unique when it exists, is called the *integral* of f over A, denoted by $\int_A f$.

Let A be a BV set and let $\mathcal{I}(A)$ denote the set of all integrable functions in A. The following statements have been proved in [$\mathbf{P}_1$], except for statement (7) proved in [**B**].

(1) The integrability of a function $f : A^* \to \mathbf{R}$ in A as well as the value of $\int_A f$ depends only on f restricted to $E \subset A$ with $\lambda(A - E) = 0$.

(2) LINEARITY. The family $\mathcal{I}(A)$ is a linear space, and the map $f \mapsto \int_A f$ is a nonnegative linear functional on $\mathcal{I}(A)$.

(3) ADDITIVITY. If $f \in \mathcal{I}(A)$ then f is integrable in B for each BV set $B \subset A$. The map $\int f : B \mapsto \int_B f$, called the *indefinite integral* of f in A, is an additive function on BV subsets of A.

(4) CONTINUITY. If $f \in \mathcal{I}(A)$ then $\int f$ is continuous in the following sense: given $\varepsilon > 0$, there is a $\kappa > 0$ such that $|\int_B f| < \varepsilon$ for each BV set $B \subset A$ for which $\|B\| < 1/\varepsilon$ and $\lambda(B) < \kappa$.

(5) λ-almost everywhere in A, each $f \in \mathcal{I}(A)$ is a derivate of $\int f$. In particular, each $f \in \mathcal{I}(A)$ is λ-measurable.

(6) A function f on A is Lebesgue integrable in A with respect to λ if and only if both f and $|f|$ are integrable in A, in which case $\int_A f = (L)\int_A f \, d\lambda$.

(7) NONTRIVIALITY. If $m \geq 2$ and the topological interior of A is nonempty, then there is a function $f \in \mathcal{I}(A)$ which is not Lebesgue integrable (with respect to λ) on any nonempty open subset of A.

(8) THE GAUSS-GREEN THEOREM. Let $T \subset A^*$ be a countable union of sets whose $\mathcal{H}$ measure is finite, and let v be a vector field defined in an open set containing the topological closure of A. Suppose that v is continuous in the topological closure of A and that

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty$$

for each $x \in A^* - T$. Then $\operatorname{div} v$, defined λ-almost everywhere in A by Stepanoff's theorem ([**F**, Theorem 3.1.9]), is integrable in A and

$$\int_A \operatorname{div} v = (L) \int_{\partial^* A} v \cdot n_A \, d\mathcal{H}$$

where n_A is the Federer exterior normal of A defined in [**F**, Section 4.5.5].

(9) COORDINATE INVARIANCE. Let $\Phi : A \to \mathbf{R}^m$ be a lipeomorphism (i.e., a bi-Lipschitzian map) and let $\det \Phi$ be the determinant of the differential of Φ (defined

λ-almost everywhere in A). Then $\Phi(A)$ is a BV set, and for each function f integrable in $\Phi(A)$ the function $f \circ \Phi \cdot |\det \Phi|$ is integrable in A and

$$\int_A f \circ \Phi \cdot |\det \Phi| = \int_{\Phi(A)} f \, .$$

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