Real Analysis Exchange Vol. 14 (1988-89)

F.S. Cater, Department of Mathematics, Portland State University, Portland, Oregon 97207 U.S.A.

DINI DERIVATES OF AN UNUSUAL FUNCTION

Let f be a real valued function on $[0,1]$ and let $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, $D_-f(x)$ denote the usual four Dini derivates of f at x. From [1] we see that f is nondecreasing on $[0,1]$ if f is continuous on $[0,1]$ and $D^+f(x) \geq 0$ for $0 \leq x < 1$. However some nondecreasing functions are discontinuous at countably many points. One wonders if the result cited above will work when f is allowed to be discontinuous at countably many points in $[0,1]$. Evidently not, for consider the characteristic function of $(\frac{1}{4},\frac{1}{2})$ for example. For a long while the author conjectured that f must be nondecreasing on $[0,1]$ if f is continuous at all but possibly countably many points in $[0,1]$ and $D^+f(x) \geq 0$ for $0 \leq x < 1$ and $D^-f(x) \geq 0$ for $0 < x \leq 1$. This is a persuasive conjecture because f surely must be nondecreasing when there are only finitely many points of discontinuity. In this note we prove the conjecture false. Indeed we prove:

Theorem 1. There exists a function f on $[0,1]$, not of bounded variation, satisfying $D^+f(x) \geq 0 \geq D_+f(x)$ for $0 \leq x < 1$ and $D^-f(x) \geq 0 \geq D_-f(x)$ for $0 < x \leq 1$, such that f is continuous at all but at most countably many points.

We can achieve montonicity by increasing the hypothesis slightly.

Proposition 1. A function f is nondecreasing on $[0,1]$ if and only if (i) and (ii) hold:

(i) f is continuous at all but a most countably many points and

$$\Sigma_x \left(\lim_{h\searrow 0} \sup\{f(t): x-h<t<x+h\} - \lim_{h\searrow 0} \inf\{f(t): x-h<t<x+h\}\right) < \infty$$

where the sum is taken over all the points of discontinuity of f,

(ii) the upper Dini derivates of f satisfy $D^-f(x) \geq 0$ for $0 < x \leq 1$ and $D^+f(x) \geq 0$ for $0 \leq x < 1$.

The proof of Proposition 1 is almost routine, so we will only sketch the proof here.

Proof of Theorem 1. We will construct the function f.

Let g be a function defined on a doubleton set $\{a,b\}$, $a < b$. For each positive integer n, define the mutually disjoint intervals

$$I_n = [\tfrac{1}{2}(b-a)(1-(4n)^{-1}) + \tfrac{1}{2}(b+a),\ \tfrac{1}{2}(b-a)(1-(4n+1)^{-1}) + \tfrac{1}{2}(b+a)]\ ,$$

$$I_{-n} = [\tfrac{1}{2}(b-a)((4n+1)^{-1}-1) + \tfrac{1}{2}(b+a),\ \tfrac{1}{2}(b-a)((4n)^{-1}-1) + \tfrac{1}{2}(b+a)]\ .$$

Also put

$$I_0 = [-\tfrac{1}{4}(b-a) + \tfrac{1}{2}(b+a),\ \tfrac{1}{4}(b-a) + \tfrac{1}{2}(b+a)]\ .$$

Define the even extension of g to be the function $\bar{g}$ on $\{a,b\} \cup I_0$ such that $\bar{g}(a) = g(a)$, $\bar{g}(b) = g(b)$, and $\bar{g} = \tfrac{1}{2}(g(a) + g(b))$ on I_0. Define the odd extension of g to be the function $\bar{g}$ on $\{a,b\} \cup \bigcup_{j=-\infty}^{\infty} I_j$ such that $\bar{g}(a) = g(a)$, $\bar{g}(b) = g(b)$, $\bar{g} = g(a)$ on I_j for j odd, and $\bar{g} = g(b)$ on I_j for j even.

More generally, if g is any function defined on the set E, let the even (odd) extension of g be the common extension of g together with the even (odd) extensions of all the restrictions of g to doubleton sets $\{a,b\}$ (if any) where a and b are consecutive points in E.

Now put $g_0(0) = 0$ and $g_0(1) = 1$. Let g_1 be the odd extension of g_0, g_2 the even extension of g_1, g_3 the odd extension of g_2, etc. In general, g_j is the even (odd) extension of g_{j-1} if j is even (odd). Let E_j denote the domain of g_j. Let g be the common extension of all the g_j on $\bigcup_{j=0}^{\infty} E_j = E$. Then E is dense in $[0,1]$.

Note that g_{2n+1} (and g_{2n}) differ on consecutive intervals of $E_{2n+1}(E_{2n})$ exactly half as much as g_{2n-1} differs on consecutive intervals of E_{2n-1}. Thus if $x \in [0,1]\setminus E$, then $\lim_{t\to x,\ t\in E} g(t)$ exists. Define $f(x) = \lim_{t\to x,\ t\in E} g(t)$ for $x \in [0,1]\setminus E$, and $f(x) = g(x)$ for $x \in E$. Then f is defined on $[0,1]$ and the only points at which f can be discontinuous are the endpoints of the component intervals of E. Thus f is continuous at all but countably many points. Moreover, from the definition of g_1 we see that f is not of bounded variation on $[0,1]$.

It remains only to prove the inequalities for the Dini derivates of f. First let x be a right endpoint of some component interval of E_j for j even. From the definition of g_{j+1}, it follows that x is a right accumulation point of the set $f^{-1}f(x)$, so $D^+f(x) \geq 0 \geq D_+f(x)$. Likewise $D^+f(0) \geq 0 \geq D_+f(0)$. Next let $x \in [0,1]\setminus E$. Say x lies between consecutive intervals I and J of E_j (j even, $I < J$). Then $f(x)$ lies between $f(I)$ and $f(J)$. We see from the definition of g_{j+1} that f assumes the values $f(I)$ and $f(J)$ at some points between x and J. Hence $D^+f(x) \geq 0 \geq D_+f(x)$. Finally $D^+f(x) \geq 0 \geq D_+f(x)$ for all x satisfying $0 \leq x < 1$. The other inequality is proved analogously. □

It can also be shown in our construction that f is either left or right continuous (or both) at each point of $(0,1)$. Moreover, f has zero derivative in the interior of each interval in E. Thus f is almost everywhere differentiable on $(0,1)$.

Sketch of the proof of Proposition 1. The necessity of (i) and (ii) is clear, so we prove sufficiency. Assume (i) and (ii). Let $x_1, x_2, x_3, \ldots, x_n, \ldots$ be the points of discontinuity of f enumerated and let

$$u_m = \lim_{h\searrow 0} \sup\{f(t)\colon x_m-h<t<x_m+h\} - \lim_{h\searrow 0} \inf\{f(t)\colon x_m-h<t<x_m+h\}$$

for each $m \geq 1$. Put

$$g_n(x) = \Sigma_{j>n,\ x_j<x}\, u_j - \Sigma_{j>n,\ x_j>x}\, u_j$$

for $0 \leq x \leq 1$ and any positive integer n, and put $f_n(x) = f(x) + g_n(x)$. Then f_n converges uniformly to f and it suffices to prove that each f_n is nondecreasing on $[0,1]$.

Let (a,b) be an open subinterval of $[0,1]$ that contains no point $x_1,\ldots,x_n$. It is not difficult to show that

$$\limsup_{h\searrow 0} f_n(x-h) \leq f_n(x) \leq \limsup_{h\searrow 0} f_n(x+h)$$

for any x in (a,b). Moreover g_n is nondecreasing so $D^+f_n \geq D^+f \geq 0$ on (a,b). Finally, f_n is nondecreasing on (a,b) by [1].

Thus it follows that $[0,1]$ can be partitioned into $n+1$ subintervals such that f_n is nondecreasing on the interior of each subinterval. From $D^+f \geq 0$ and $D^-f \geq 0$, it can be shown that f_n is nondecreasing on $[0,1]$.

REFERENCE

1. S. Saks, Theory of the Integral, Revised Second Edition, Dover Publications, New York, 1964, Theorem (7.2), p. 204.

Received June 16, 1988