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Answers to three questions of Foran

In [3] James Foran asked 12 questions. Here we give answers to the following three:

Question 7. ([3],p.99) How may the functions of the form $f\circ g$, where f is an absolutely continuous homeomorphism and g is differentiable, be characterized?

Question 6. ([3],p.98) How can the class of functions of the form $f\circ g$, where f is a homeomorphism and g satisfies Banach's condition T_2 and is continuous, be characterized?

Question 1. ([3],p.92) Can every ACG_* and continuous function be written as $f\circ g$, where f is differentiable and g is monotone and absolutely continuous?

Theorem 1, Theorem 2 and Theorem 3 are answers to these three questions.

In what follows all functions are supposed to be defined on $[0,1]$ with range in $[0,1]$. Let $H = \{h : h$ is a homeomorphism which maps $[0,1]$ onto $[0,1]\}$; $\overline{H} = \{h : h\in H\cap AC\}$; $\overline{\overline{H}} = \{h : h\in H\cap AC$ and $h^{-1}\in H\cap AC\}$; diff $= \{f : f$ is differentiable$\}$; bdiff $= \{f : f$ is differentiable and f' is bounded$\}$; $L = \{f : f$ is a Lipschitz function$\}$; $\mathcal{C} = \{f : f$ is continuous$\}$. For Banach's conditions (S) and T_2, for ACG_* and AC see [4]. We need also the following facts:

Theorem A. ([1],p.202) If $h\in \overline{H}$ and $h'(x)\neq 0$ a.e. then $h\in \overline{\overline{H}}$.

Observation. In Theorem A we can not give up the condition

" $h'(x) \neq 0$ a.e. ". Indeed, there are $h \in \overline{H}$ with $h'(x) \geq 0$ a.e. but $h'(x) = 0$ on a set of positive measure. For such h, $h^{-1} \notin AC$ and $h \notin \overline{\overline{H}}$.

Theorem B. If $F \in L$ then there exists $h \in \overline{\overline{H}} \cap$ bdiff such that $h \circ F \in$ bdiff.

Proof. Let $F \in L$. By [2],p.134 there ecists $h \in \overline{H} \cap$ bdiff with $h'(x) > 0$ a.e. and $h \circ F \in$ bdiff. Clearly by Theorem A, $h \in \overline{\overline{H}}$.

Theorem C. ([4],p.237) Let $F \in \mathcal{C}$. $F \in VBG_*$ (resp. $ACG_* \cap \mathcal{C}$) if and only if there exists a continuous, increasing function U which maps [0,1] onto itself (resp. U is increasing and AC) such that the extreme derivatives of F with respect to U are finite at each point of [0,1]. (Without loss of generality we may suppose $U'(x) > 0$ a.e., if not we can replace $U(x)$ with $U_1(x) = (U(x)+x)/2$, so by Theorem A, $U \in \overline{H}$ (resp. $U \in \overline{\overline{H}}$).)

Theorem D. A function F satisfies a Lipschitz condition with constant $M > 0$ if and only if the Dini derivatives of F are bounded in absolute value by M.

Proof. The necessity of this condition is evident. Suppose that the Dini derivatives of F are bounded in absolute value by M. Then by [4](Lemma 6.3,p.226), $|F((x_1,x_2))| \leq M \cdot |x_1 - x_2|$, $0 \leq x_1 < x_2 \leq 1$ Since F is continuous it follows that $|F(x_1) - F(x_2)| \leq M \cdot |x_1 - x_2|$.

Theorem 1. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two classes of continuous functions such that $\overline{\overline{H}} \subseteq \mathcal{A}_1 \subseteq (S)$ and bdiff $\subseteq \mathcal{A}_2 \subseteq (S)$. Then $\mathcal{A}_1 \circ \mathcal{A}_2 = (S)$.

Remark 1. a) Clearly bdiff $\subset L \subset AC \subset ACG_* \subset (S)$ and bdiff $\subset$ diff $\subset ACG_* \subset (S)$.
b) For $\mathcal{A}_1 = \overline{H}$ and $\mathcal{A}_2 =$ diff we obtain the answer to Question 7.

__Proof of Theorem 1.__ By [1](p.209) it follows that $(S) = AC \circ AC$. It follows from the proof of Nina Bary's theorem ([1],pp.203–207) and Theorem D that the composition of two AC functions can be ones whose outer function is increasing and AC with an inverse that is a Lipschitz function, and whose inner function is a Lipschitz function. (The statement of the theorem asserts only that the outer function be increasing and absolutely continuous and that the inner function be absolutely continuous.) Hence $(S) = AC \circ AC = \bar{\bar{H}} \circ L$. By Theorem B, $L \subset \bar{\bar{H}} \circ \mathrm{bdiff}$. Since $\bar{\bar{H}} \circ \bar{\bar{H}} = \bar{\bar{H}}$ and $\mathrm{bdiff} \subset L$ it follows that $(S) = \bar{\bar{H}} \circ L \subseteq \bar{\bar{H}} \circ \bar{\bar{H}} \circ \mathrm{bdiff} = \bar{\bar{H}} \circ \mathrm{bdiff} \subseteq \bar{\bar{H}} \circ L = (S)$. Hence $(S) = \bar{\bar{H}} \circ \mathrm{bdiff}$. Now we prove that $\bar{\bar{H}} \circ (S) = (S)$. Indeed, $(S) \subseteq \bar{\bar{H}} \circ (S) = \bar{\bar{H}} \circ \bar{\bar{H}} \circ L = \bar{\bar{H}} \circ L = (S)$. We show that $(S) \circ (S) = (S)$ (this follows also by [1], pp.214–216). It is easy to verify that $AC \circ \bar{H} = AC$. Hence $(S) \subseteq (S) \circ (S) = \bar{\bar{H}} \circ L \circ \bar{\bar{H}} \circ L \quad \bar{\bar{H}} \circ AC \circ \bar{\bar{H}} \circ AC = \bar{\bar{H}} \circ AC \circ AC = \bar{\bar{H}} \circ (S) = (S)$ and $(S) = \bar{\bar{H}} \circ \mathrm{bdiff} \subseteq \mathcal{A}_1 \circ \mathcal{A}_2 \subseteq (S) \circ (S) = (S)$. It follows that $\mathcal{A}_1 \circ \mathcal{A}_2 = (S)$.

__Theorem 2.__ $VBG_* \cap \mathcal{C} = \mathrm{diff} \circ H$ __and__ $ACG_* \cap \mathcal{C} = \mathrm{diff} \circ \bar{H}$.

__Proof.__ Clearly $\mathrm{diff} \circ H \subseteq (ACG_* \cap \mathcal{C}) \circ H \subseteq (VBG_* \cap \mathcal{C}) \circ H = VBG_* \cap \mathcal{C}$ and $\mathrm{diff} \circ \bar{\bar{H}} \subseteq (ACG_* \cap \mathcal{C}) \circ \bar{\bar{H}} \subseteq (VBG_* \cap \mathcal{C}) \cap (N) = ACG_* \cap \mathcal{C}$. Conversely it is sufficient to show that if $F \in \mathcal{C} \cap VBG_*$ (resp. $\mathcal{C} \cap ACG_*$) then there exists $h \in H$ (resp. $\bar{\bar{H}}$) such that $F \circ h$ is differentiable. Let U be defined as in Theorem C then $F \circ U^{-1} = F_1$ has finite extreme derivatives at each point of $[0,1]$. By [4](Theorem 10.5,o.235), $F_1 \in ACG_*$ on $[0,1]$. Hence F_1 is differentiable a.e. on $[0,1]$. Let $W = \{x : F_1$ is not differentiable at $x\}$. Then $|W| = 0$. Let Z be a G_δ-set of measure 0 such that $W \subset Z \subset [0,1]$. By [2], p.126 it follows that there exist $h^{-1} \in \bar{H}$ and $a > 0$ such that $(h^{-1})'(x) = \infty$. if $x \in Z$ and $(h^{-1})'(x) > a$, if $x \in [0,1] - Z$. By Theorem A it follows that $h^{-1} \in \bar{\bar{H}}$. Then $h'(x) = 0$ for $x \in h^{-1}(Z)$

and $h'(x) < 1/a$ for $x \in [0,1] - h^{-1}(Z)$. Following Bruckner ([3],p. 126) we obtain that $(F_1 \circ h)'(x) = F_1'(h(x)) \circ h'(x)$ if $h(x) \notin Z$ and $(F_1 \circ h)'(x) = 0$ if $h(x) \in Z$. Let $g = U^{-1} \circ h$ then $g \in H$ (resp. $\bar{\bar{H}}$) and $F \circ g \in$ diff.

Remark 2. a) The first part of Theorem 2 is identical with Theorem 1.5 ([2],p.129) proved in another way by Fleissner and Foran; b) The second part of Theorem 2 is the answer to Question 1.

Definition. A continuous function f satisfies condition B_3 on $[0,1]$ if the set $\{y : f^{-1}(y)$ is at most countable$\} \cap J$ contains a perfect subset for each nondegenerate closed interval $J \subseteq f([0,1])$.

Theorem 3. For continuous functions $H \circ T_2 = B_3$.

Proof. Let $h \in H$ and $f \in T_2$. We may suppose without loss of generality that f maps $[0,1]$ onto itself. Let $F = h \circ f$, $E = \{y : f^{-1}(y)$ is at most countable$\}$; $E_1 = \{z : F^{-1}(z)$ is at most countable$\}$ Then $E_1 = h(E)$. Let J be a closed nondegenerate subinterval of $[0,1]$ and let $I = h^{-1}(J)$. Since $f \in T_2$ it follows that $m(E) = m(E \cap I)$. Let P be a perfect subset of $E \cap I$ then $h(P) = Q$ is a perfect subset of $J \cap E_1$, hence $F \in B_3$. Conversely, let $F \in B_3$. We may suppose without loss of generality that F maps $[0,1]$ onto itself. Let $E = \{y : F^{-1}(y)$ is at most countable$\}$. Since $F \in B_3$, if $\{I_n\}_n$ denotes an enumeration of those subintervals of $[0,1]$ which have rational end-points we can obtain a sequence of nowhere dense perfect sets Q_n such that for each n, $Q_n \subset I_n \cap E$. Let $Q = \bigcup_{n=1}^{\infty} Q_n$. Clearly $Q \subset E$ and Q is a c-dense subset of $[0,1]$, of F_σ-type. By [2](Lemma 1.7,p. 129) there exists a homeomorphism h of $[0,1]$ onto itself such that $m(h(Q)) = 1$. Let $F_1 = h \circ F$, $E_1 = h(E)$. Clearly $E_1 = \{z : F^{-1}(z)$ is at most countable$\}$. Since $m(E_1) = 1$ and F_1 maps $[0,1]$ onto $[0,1]$ it follows that $F_1 \in T_2$.

Remark 3. Theorem 3 is the answer to Question 6.

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