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Some remarks on density-continuous functions

1. Introduction

Definition: The ***density topology*** on $\mathbb{R}$ consists of all Lebesgue measurable sets $E \subseteq \mathbb{R}$ such that for all $x \in E$, $\mathrm{d}(x, E) = 1$, where $\mathrm{d}(x, E) = \lim_{\delta \to 0}(1/2\delta)\mu\big(E \cap (x - \delta, x + \delta)\big)$.

The density topology is a $T_{3\frac{1}{2}}$ connected refinement of the Euclidean topology (see e.g. [**Tall**]).

Definition: If $f: \mathbb{R} \to \mathbb{R}$ we say f is ***density-continuous*** if f is continuous as a self-map of $\mathbb{R}$, when $\mathbb{R}$ is given the density topology.

Several people have considered the properties of such functions. (See [**Bru**], [**Ost 1**], [**Ost 2**], [**Ost 3**], [**Ost 4**], [**Nie**].) The following questions are asked in [**Ost 4**]:

1. Are polynomials density-continuous?

2. Does there exist a density-continuous function f such that the function $x \mapsto f(x)+x$ is not density-continuous?

Here we prove the folowing:

Theorem 1: ***Real-analytic functions are density-continuous.***

Theorem 2: ***There is a differentiable density-continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f'(0) > 0$ and the function $x \mapsto f(x) + x$ is not density-continuous at*** 0.

2. Proof of theorem 1

Let $f: \mathbb{R} \to \mathbb{R}$ be a real-analytic function, and fix $x_0 \in \mathbb{R}$ and a measurable set $E \subseteq \mathbb{R}$ such that $f(x_0) \in E$ and $\mathrm{d}(f(x_0), E) = 1$. We must show that $\mathrm{d}(x_0, f^{-1}(E)) = 1$.

Case 1: $f'(x_0) \neq 0$, say $f'(x_0) > 0$. In this case the fact that f is C^1 implies that it is density-continuous. This follows easily from the results in [**Bru**] but for the sake of completeness we give here the following proof:

Denote by I_δ the interval $(f(x_0-\delta), f(x_0+\delta))$. Let $y_0 = f(x_0)$. Fix $\epsilon > 0$ and choose $\delta_0 > 0$ such that $f' > 0$ on $(x_0-\delta_0, x_0+\delta_0)$, for any $\delta \in (0,\delta_0)$ we have $\mu(E \cap I_\delta) \,/\, \mu(I_\delta) \geq 1-\epsilon$ and $\mu(I_\delta)\,/\,2\delta \geq (1-\epsilon)f'(x_0)$, and for any $y \in I_{\delta_0}$ we have $(f^{-1})'(y) \geq (1-\epsilon)(f^{-1})'(y_0)$.

Then for any $0 < \delta < \delta_0$,

$$\begin{aligned}
\frac{1}{2\delta}\mu(f^{-1}(E)\cap(x_0-\delta,x_0+\delta)) &= \frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta}\chi_{f^{-1}(E)}(t)\,dt = \frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta}\chi_E(f(t))\,dt\\
&= \frac{1}{2\delta}\int_{f(x_0-\delta)}^{f(x_0+\delta)}\chi_E(y)(f^{-1})'(y)\,dy\\
&\geq \frac{1}{2\delta}(1-\epsilon)(f^{-1})'(y_0)\int_{f(x_0-\delta)}^{f(x_0+\delta)}\chi_E(y)\,dy\\
&= \frac{1}{2\delta}(1-\epsilon)\frac{1}{f'(x_0)}\mu(E\cap I_\delta)\\
&\geq \frac{(1-\epsilon)f'(x_0)}{\mu(I_\delta)}(1-\epsilon)\frac{1}{f'(x_0)}\mu(E\cap I_\delta)\\
&= (1-\epsilon)^2\frac{\mu(E\cap I_\delta)}{\mu(I_\delta)} \geq (1-\epsilon)^3.
\end{aligned}$$

Thus $\mathrm{d}(x_0, f^{-1}(E)) = 1$.

Case 2: $f'(x_0) = 0$. Since Lebesgue measure and the density topology are translation invariant, we may assume that $x_0 = 0$ and $f(0) = 0$. We may also assume that f is not constant and thus that f' has no zeros other than 0 in some neighborhood of 0. Also it suffices to show that the one-sided densities of $f^{-1}(E)$ at 0 are equal to 1. Thus we have to show that

$$\lim_{\delta\to 0}\frac{\mu(f^{-1}(E)\cap(0,\delta))}{\delta} = 1.$$

Fix a positive number $\epsilon < 1$ and an integer $n \geq 1$. For some $\delta_0 > 0$ we can write

$$f(x) = ax^p(1+\sum_{k\geq 1}a_kx^k), \qquad x \in (-\delta_0, \delta_0)$$

where $p \geq 2$. Choose $\delta_0 > 0$ sufficiently small so that f' is never 0 on $(0,\delta_0)$, without loss of generality assume $f' > 0$ on $(0,\delta_0)$. Also assume that δ_0 is sufficiently small so that for

$x \in (0, \delta_0)$,

$$f(x) \geq ax^p(1-\epsilon) \qquad \text{and} \qquad f'(x) \leq apx^{p-1}(1+\epsilon).$$

Then for $y = f(x) \in f\big((0,\delta_0)\big)$,

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(x)} \geq \frac{1}{ap(1+\epsilon)} x^{1-p} \geq \frac{1}{ap(1+\epsilon)} \left(\frac{y}{a(1-\epsilon)}\right)^{(1-p)/p} \\ &= \frac{(1-\epsilon)^{(p-1)/p}}{a^{1/p}p(1+\epsilon)y^{(p-1)/p}}.\end{aligned}$$

We will also need the following:

Lemma: *If* $\lim_{\delta\to 0}(1/\delta)\mu\big(E\cap(0,\delta)\big) = 1$, *then for any* $0 \leq r < s \leq 1$, $\lim_{\delta\to 0}(1/\delta)\mu\big(E\cap(r\delta, s\delta)\big) = s - r$. ∎

For any $\delta \in (0, \delta_0)$, we have

$$\begin{aligned}\frac{\mu\big(f^{-1}(E)\cap(0,\delta)\big)}{\delta} &= \frac{1}{\delta}\int_0^\delta \chi_{f^{-1}(E)}(t)\,dt = \frac{1}{\delta}\int_0^\delta \chi_E(f(t))\,dt \\ &= \frac{1}{\delta}\int_0^{f(\delta)} \chi_E(y)(f^{-1})'(y)\,dy \\ &\geq \frac{1}{\delta}\int_0^{a\delta^p(1-\epsilon)} \chi_E(y)\frac{(1-\epsilon)^{(p-1)/p}}{a^{1/p}p(1+\epsilon)y^{(p-1)/p}}dy \\ &= \frac{(1-\epsilon)^{(p-1)/p}}{a^{1/p}p(1+\epsilon)\delta}\int_0^{a\delta^p(1-\epsilon)} \chi_E(y)\frac{1}{y^{(p-1)/p}}dy \\ &= \frac{(1-\epsilon)^{(p-1)/p}}{a^{1/p}p(1+\epsilon)\delta}\sum_{k=0}^{n-1}\int_{\frac{k}{n}a\delta^p(1-\epsilon)}^{\frac{k+1}{n}a\delta^p(1-\epsilon)} \chi_E(y)\frac{1}{y^{(p-1)/p}}dy \\ &\geq \frac{(1-\epsilon)^{(p-1)/p}}{a^{1/p}p(1+\epsilon)\delta}\sum_{k=0}^{n-1}\frac{\mu\big(E\cap(\frac{k}{n}a\delta^p(1-\epsilon), \frac{k+1}{n}a\delta^p(1-\epsilon))\big)}{\big[\frac{k+1}{n}a\delta^p(1-\epsilon)\big]^{(p-1)/p}} \\ &= \frac{1}{p(1+\epsilon)}\sum_{k=0}^{n-1}\frac{1}{(\frac{k+1}{n})^{(p-1)/p}}\frac{\mu\big(E\cap(\frac{k}{n}a\delta^p(1-\epsilon), \frac{k+1}{n}a\delta^p(1-\epsilon))\big)}{a\delta^p}.\end{aligned}$$

It follows from the lemma that

$$\begin{aligned}\underline{\lim}_{\delta\to 0}\frac{\mu\big(f^{-1}(E)\cap(0,\delta)\big)}{\delta} &\geq \frac{1}{p(1+\epsilon)}\sum_{k=0}^{n-1}\frac{1}{(\frac{k+1}{n})^{(p-1)/p}}\frac{1-\epsilon}{n} \\ &\longrightarrow \frac{(1-\epsilon)}{(1+\epsilon)}\int_0^1 \frac{dy}{py^{(p-1)/p}} = \frac{(1-\epsilon)}{(1+\epsilon)} \qquad \text{as } n\to\infty.\end{aligned}$$

Since ϵ is arbitrary,

$$\lim_{\delta\to 0} \frac{\mu(f^{-1}(E)\cap(0,\delta))}{\delta} = 1,$$

as desired. ∎

2. Proof of theorem 2 $f\colon \mathbb{R} \to \mathbb{R}$ is constructed as follows: $f(x) = x$ if $x \le 0$, $f(x) = 1-x$ if $x \ge \frac{1}{2}$, and in each interval $[\frac{1}{2n}, \frac{1}{2(n-1)}]$, $n \ge 2$, we define $f_n(x) = f(x) = \frac{1}{n} - x$, if $\frac{1}{2n} \le x \le \frac{1}{2(n-1)} - 2^{-n-10}$, and f is linear and continuous on $[\frac{1}{2(n-1)} - 2^{-n-10}, \frac{1}{2(n-1)}]$. See figures 1 and 2.

Let $E = \mathbb{R} \setminus \{1/n : n = 1,2,\ldots\}$. Then $\mathrm{d}(0,E) = 1$, however

$$\{x \in \mathbb{R} : f(x) + x \in E\} \subseteq (-\infty, 0) \cup \bigcup_{n=2}^{\infty} [\tfrac{1}{2(n-1)} - 2^{-n-10}, \tfrac{1}{2(n-1)}],$$

and this union has density 0 to the right of 0. Thus the map $x \mapsto f(x) + x$ is not density-continuous.

On the other hand, for any set S which has density 1 at 0,

$$\begin{aligned} f^{-1}(S) \cap (0, \tfrac{1}{2(n-1)}) &\supseteq \bigcup_{k\ge n} f^{-1}(S) \cap (\tfrac{1}{2k}, \tfrac{1}{2(k-1)}) \\ &\supseteq \bigcup_{k\ge n} f_k^{-1}(S \cap (\tfrac{1}{2(k+1)}, \tfrac{1}{2k})); \end{aligned}$$

since this union is disjoint and the f_k's are measure-preserving, we have

$$\begin{aligned} \mu(f^{-1}(S) \cap (0, \tfrac{1}{2(n-1)})) &\ge \sum_{k\ge n} \mu(S \cap (\tfrac{1}{2(k+1)}, \tfrac{1}{2k})) \\ &= \mu(S \cap (0, \tfrac{1}{2n})), \end{aligned}$$

and thus

$$\begin{aligned} \frac{\mu(f^{-1}(S) \cap (0, \frac{1}{2(n-1)}))}{1/(2(n-1))} &\ge \frac{\mu(S \cap (0, \frac{1}{2n}))}{1/(2(n-1))} \\ &= \frac{2(n-1)}{2n} \frac{\mu(S \cap (0, \frac{1}{2n}))}{(1/2n)} \\ &\longrightarrow 1 \text{ as } n \to \infty. \end{aligned}$$

Thus f is density-continuous at 0. Clearly f is density-continuous at every other point as well. Also note that f is differentiable at 0: see figure 2. It is straightforward (cf. figure

2, inset) to modify the graph of f in small neighborhoods of the points where it is not differentiable, replacing it with a suitable polynomial of degree 2, to make f differentiable everywhere while preserving the density-continuity of f (theorem 1 is needed here) and the discontinuity at 0 of the map $x \mapsto f(x) + x$. ∎

Remarks:

1. W. Just and independently K. Ciesielski and L. Larson constructed a C^∞ not density-continuous function $g: \mathbb{R} \to \mathbb{R}$ whose derivative is bounded. It is easy to see that for a sufficiently small positive constant c, the function f defined by $f(x) = cg(x) - x$ will be density-continuous and that $f(x) + x$ will not be density-continuous. Of course, since f is C^∞, $f(x) + x$ has derivative zero at the point where it fails to be density-continuous.

2. K. Ciesielski and L. Larson have independently proven theorem 1, by different methods.

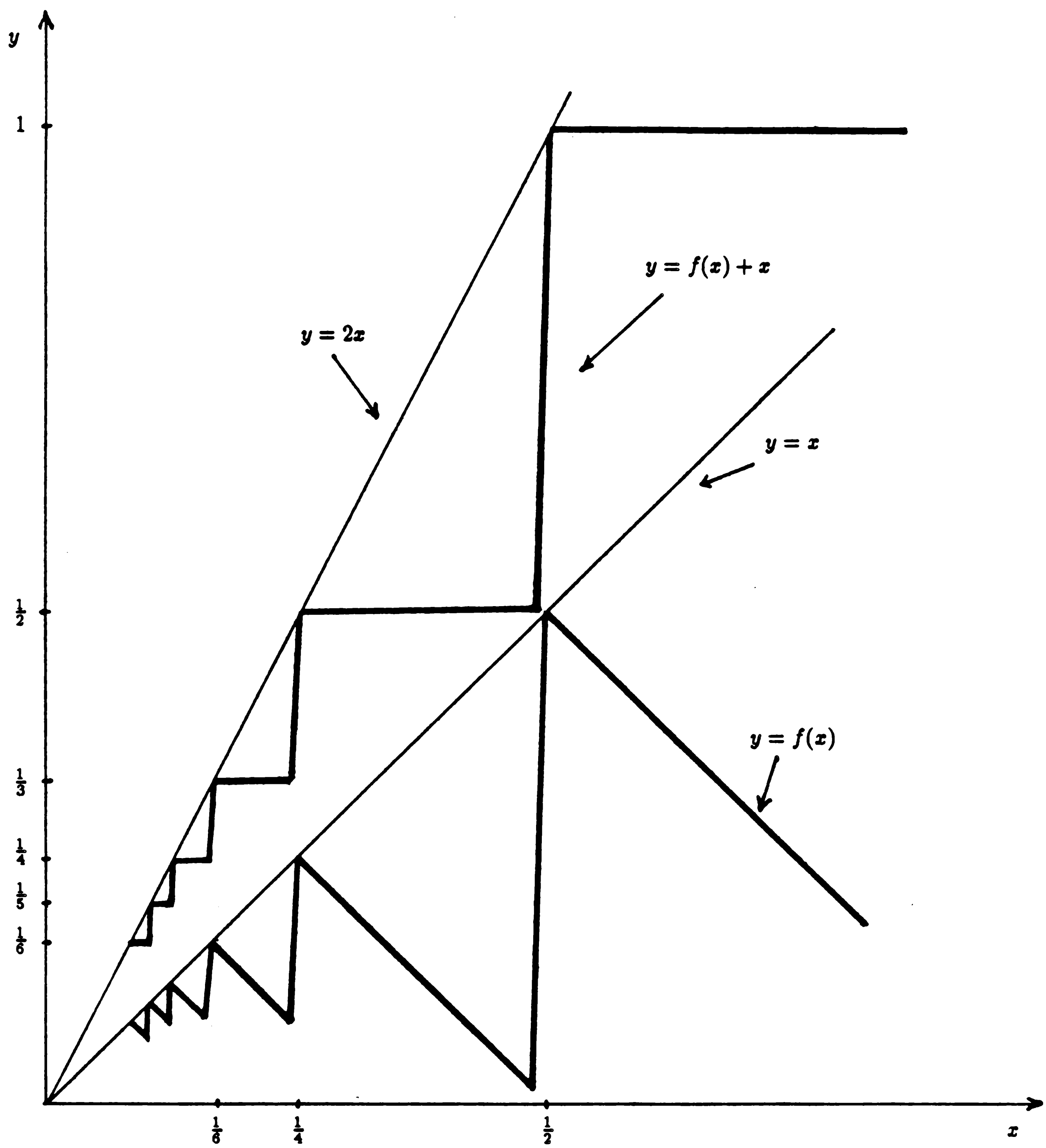
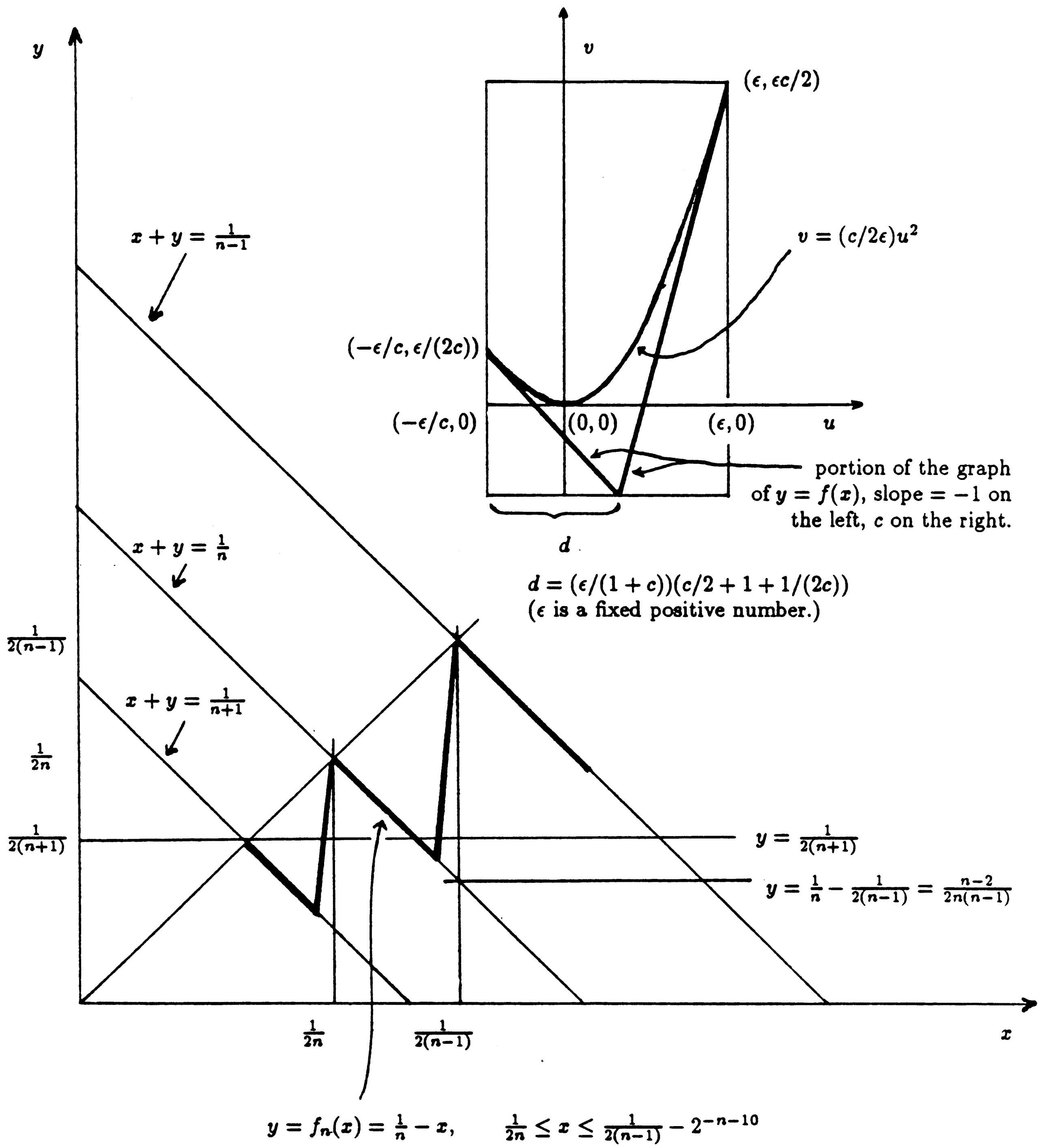


Figure 1: Graphs of $f(x)$ and $f(x) + x$.



Remarks: 1. Since $\frac{n-2}{2n(n-1)} + 2^{-n-10} < \frac{1}{2(n+1)}$, f_n maps *onto* $[\frac{1}{2(n+1)}, \frac{1}{2n}]$.

2. For $x \in [\frac{1}{2n}, \frac{1}{2(n-1)}]$, the slope $f(x)/x$ has a minimum value of $\frac{n-2}{n}$ at $x = \frac{1}{2(n-1)}$.
As $n \to \infty$, $\frac{n-2}{n} \to 1$. Thus $f'(0) = 1$.

Figure 2: Details of the definition of $f(x)$.

Inset: Modification which makes f differentiable at the points $x = \frac{1}{2(n-1)} - 2^{-n-10}$.
The points $x = \frac{1}{2n}$ are handled similarly.

References:

[**Bru**] Bruckner A.M., *Density-preserving homeomorphisms and a theorem of Maximoff*, Quart. J. Math. Oxford, (2) 21 (1970) 337–347.

[**Nie**] Niewiarowski J., *Density preserving homeomorphisms*, Fund. Math., 106 (1980) 77–87.

[**Ost 1**] Ostaszewski K., *Continuity in the density topology*, Real Anal. Exchange, 7 (1981–82) 259–270.

[**Ost 2**] Ostaszewski K., *Continuity in the density topology II*, Rend. Circ. Mat. Palermo, 32 (1983) 398–414.

[**Ost 3**] Ostaszewski K., *Density topology and the Luzin (N) condition*, Real Anal. Exchange, 9 (1983–84) 390–393.

[**Ost 4**] Ostaszewski K., *Semigroups of density-continuous functions*, preprint.

[**Tall**] Tall F.D., *The density topology*, Pac. J. Math., 62 (1976) 275–284.

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