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WHEN IS THE INTEGRATION ON QUANTUM PROBABILITY SPACES ADDITIVE?

1. Introduction. In the paper [9] S. Gudder and J. Zerbe proved that for finitely valued functions the integral on quantum probability spaces is additive. Their proof involved highly nontrivial combinatorial reasoning. By applying a new (essentially plane topological) method, we extend the additivity result to a broad class of functions. (See Theorem 6.) When specialized to finitely valued functions, we also provide a simpler proof of the original Gudder-Zerbe theorem.

Following [3], a couple (X, L) is called a *σ-class* if X is a nonempty set and $L \subset 2^X$ such that

(i) $\emptyset \in L$,

(ii) if $A \in L$, then $X - A \in L$,

(iii) if $A_i \in L$ $(i \in N)$ are mutually disjoint, then $\bigcup_{i \in N} A_i \in L$.

Further, a function $s : L \to [0, 1]$ is called a *state* if $s(\emptyset) = 0$ and $s(\bigcup_{i \in N} A_i) = \sum_{i \in N} s(A_i)$ for every mutually disjoint sequence $A_i \in L$ $(i \in N)$. The triple (X, L, s), where (X, L) is a σ-class and s is a state on L, is called a *quantum probability space* (abbr. QPS).

Quantum probability spaces - besides being a formal generalization of classical probability spaces - naturally appeared as "event structures" in theories involving the phenomenon of "compatibility" [2, 6, 7, 8]. Here we shall be interested exclusively in the so called additivity problem.

Its motivation comes from quantum mechanics (See [2, 3].)

Let (X, L, s) be a QPS and let $f: X \to R$ be a function measurable with respect to L. Then the collection $A = f^{-1}(\mathcal{B}(R))$, where $\mathcal{B}(R)$ is the σ-algebra of all Borel subsets of R, is a Boolean σ-algebra contained in L. If we now restrict the state s to A we obtain an ordinary (probability) measure. The integral $\int f\, ds$ then may (and will) be understood as the Lebesgue integral with respect to s on A.

Let f, g be two bounded measurable functions on (X, L, s) and suppose that $f + g$ is measurable,too. Do we then have the equality

$$\int f\, ds + \int g\, ds = \int (f + g)\, ds \ ?$$

Since in every integral we view the state s as an ordinary probability measure on the respective σ-algebras generated by f, g and $f + g$, all the integrals obviously converge. (Also see [3] for relevant comments.) There are examples showing that this equality does not have to hold in general [1, 4, 5]. On the other hand, in [9] the authors showed that the equality holds provided both f, g are finitely valued. Their proof was nontrivial and quite intriguing. Here we intend to offer another method (more transparent in the opinion of the author) which allows a principal extension of the Gudder-Zerbe theorem.

2. Main result. Prior to proving our result, let us assume that throughout the paper f, g and $f + g$ are fixed bounded measurable functions on a QPS (X, L, s). Without any loss of generality we may (and shall) assume that L is the σ-class generated by $f^{-1}(\mathcal{B}(R)) \cup g^{-1}(\mathcal{B}(R)) \cup (f + g)^{-1}(\mathcal{B}(R))$ (i. e., L is the least σ-class on which f, g and $f + g$ are all measurable).

Let R_+ denote the set of all positive real numbers.

Define, for any $\varepsilon \in R_+$, the functions $f_\varepsilon = \varepsilon \operatorname{int}(f/\varepsilon)$, $g_\varepsilon =$

$= \varepsilon\, \mathrm{int}(g/\varepsilon)$ and $h_\varepsilon = \varepsilon\, \mathrm{int}((f + g)/\varepsilon)$, where int stands for the greatest integer part function. If $\varepsilon \to 0_+$, then the family f_ε converges uniformly to f. Similarly, we have $g_\varepsilon \to g$ and $h_\varepsilon \to f + g$. For every $\varepsilon \in R_+$ the function $1/\varepsilon\,(h_\varepsilon - (f_\varepsilon + g_\varepsilon)) = \mathrm{int}(f/\varepsilon + g/\varepsilon) -$ $- (\mathrm{int}(f/\varepsilon) + \mathrm{int}(g/\varepsilon))$ attains only the values 0 and 1. As one sees easily, the latter function is the characteristic function of a subset of X. Let us denote this set by K_ε. (The set is portrayed in Fig. 1.)

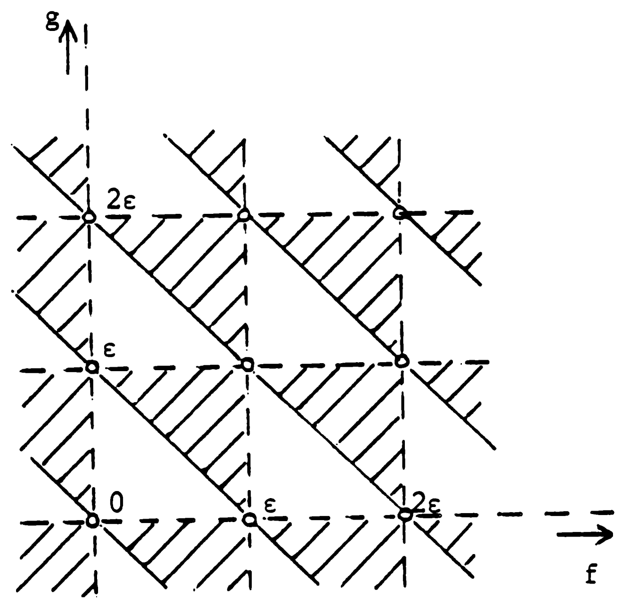

Fig. 1

We shall prove now that, under suitable assumptions, the sets K_ε $(\varepsilon \in R_+)$ belong to L and, moreover, they satisfy the following condition:

(A) $\quad s(K_\varepsilon) = 1/\varepsilon\,\left(\int h_\varepsilon\, ds - \int f_\varepsilon\, ds - \int g_\varepsilon\, ds\right)$.

Let us denote by E the set of all positive ε such that $K_\varepsilon \in L$ and the condition (A) holds.

<u>Lemma 1:</u> If the functions f, g are nonnegative then $E \neq \emptyset$.

Proof: If we take ε such that $\varepsilon > \sup f + \sup g$, we obtain $K_\varepsilon =$ $= \emptyset \in L$ and $f_\varepsilon = g_\varepsilon = h_\varepsilon = 0$. Hence the condition (A) holds.

<u>Lemma</u> 2 ("contraction"): Let the functions f, g be nonnegative and let δ, ε be nonnegative real numbers with $\delta < \varepsilon$. Suppose that

$$f^{-1}([i\varepsilon - (i+j)(\varepsilon-\delta),\ i\varepsilon)) \cap g^{-1}([j\varepsilon - (i+j)(\varepsilon-\delta),\ j\varepsilon)) = \emptyset$$

for all $i, j \in N$. Then $\delta \in E$ if and only if $\varepsilon \in E$ and, moreover, if $E \cap [\delta, \varepsilon] \neq \emptyset$, then $[\delta, \varepsilon] \subset E$.

Proof: Put, for any $\eta \in R_+$ and any $i, j \in N$, $S_\eta^{i,j} = f^{-1}([i\eta - \eta, i\eta)) \cap g^{-1}([j\eta - \eta, j\eta))$. Then $X = \bigcup_{i,j\in N} S_\eta^{i,j}$ and $S_\eta^{i,j} \cap K_\eta = S_\eta^{i,j} \cap (f+g)^{-1}([(i+j-1)\eta, (i+j)\eta))$. (See Fig. 1.) According to our assumption, $S_\varepsilon^{i,j} = \emptyset$ whenever $(i+j)(\varepsilon-\delta) \geq \varepsilon$.

The sets $(f+g)^{-1}([k\delta, k\varepsilon))$ $(k \in N)$ are mutually disjoint. (Indeed, for $k(\varepsilon-\delta) < \varepsilon$ we have $[(k-1)\delta, (k-1)\varepsilon) \cap [k\delta, k\varepsilon) = \emptyset$ and for $k(\varepsilon-\delta) \geq \varepsilon$ we obtain $(f+g)^{-1}([k\delta, k\varepsilon)) = \emptyset$.)

We shall prove now that the set $\bigcup_{k\in N}(f+g)^{-1}([k\delta, k\varepsilon)) \in L$ is disjoint from K_ε. Fro any $i, j \in N$, we have the equalities

$$\begin{aligned} &S_\varepsilon^{i,j} \cap K_\varepsilon \cap \bigcup_{k\in N}(f+g)^{-1}([k\delta, k\varepsilon)) = \\ &= (S_\varepsilon^{i,j} \cap K_\varepsilon) \cap \bigcup_{k\in N}(S_\varepsilon^{i,j} \cap (f+g)^{-1}([k\delta, k\varepsilon)) \subset S_\varepsilon^{i,j} \cap \\ &\quad \cap (f+g)^{-1}([(i+j)\delta, (i+j)\varepsilon)) \subset f^{-1}([i\varepsilon - (i+j)(\varepsilon-\delta), i\varepsilon) \cap \\ &\quad \cap g^{-1}([j\varepsilon - (i+j)(\varepsilon-\delta), j\varepsilon)) = \emptyset . \end{aligned}$$

Thus, $K_\varepsilon \cap \bigcup_{k\in N}(f+g)^{-1}([k\delta, k\varepsilon)) = \emptyset$ and the set

$$P = K_\varepsilon \cup \bigcup_{k\in N}(f+g)^{-1}([k\delta, k\varepsilon))$$

belongs to L if and only if $K_\varepsilon \in L$.

Similarly, the sets $f^{-1}([k\delta, k\varepsilon))$, $g^{-1}([k\delta, k\varepsilon))$ $(k \in N)$ are mutually disjoint. We shall prove that $\bigcup_{k\in N} f^{-1}([k\delta, k\varepsilon)) \cup \bigcup_{k\in N} g^{-1}([k\delta, k\varepsilon))$ is disjoint from K_δ. For all $i, j \in N$, we have the equalities

$$\begin{aligned} &S_\delta^{i,j} \cap K_\delta \cap \bigcup_{k\in N} f^{-1}([k\delta, k\varepsilon)) = \\ &= S_\delta^{i,j} \cap (f+g)^{-1}([(i+j-1)\delta, (i+j)\delta)) \cap \\ &\quad \cap f^{-1}([(i-1)\delta, (i-1)\varepsilon)). \end{aligned}$$

The latter set is empty because it is a subset of the set

$$f^{-1}([i_1\varepsilon - (i_1+j)(\varepsilon-\delta), i_1\varepsilon)) \cap g^{-1}([j\varepsilon - (i_1+j)(\varepsilon-\delta), j\varepsilon)),$$

where $i_1 = i - 1$.

Hence $K_\delta \cap \bigcup_{k\in N} f^{-1}([k\delta, k\varepsilon)) = \emptyset$ and, analogously, $K_\delta \cap \bigcup_{k\in N} g^{-1}([k\delta, k\varepsilon)) = \emptyset$. Therefore the set

$$Q = K_\delta \cup \bigcup_{k\in N} f^{-1}([k\delta, k\varepsilon)) \cup \bigcup_{k\in N} g^{-1}([k\delta, k\varepsilon))$$

belongs to L if and only if $K_\delta \in L$.

A routine verification gives that $P = Q$. Moreover, we obtain the equality

$$s(K_\varepsilon) + \sum_{i\in N} s[(f+g)^{-1}([i\delta, i\varepsilon))] =$$
$$= s(K_\delta) + \sum_{i\in N} s[f^{-1}([i\delta, i\varepsilon))] + \sum_{i\in N} s[g^{-1}([i\delta, i\varepsilon))] ,$$

which ensures the validity of the condition (A). The rest of Lemma 2 is straightforward.

In what follows, let H denote the closure of the set $\{(f(x), g(x)) \in R^2 : x \in X\}$. (As we shall not refer to any open intervals in this paper, the symbol (p, q) always denotes an element of R^2.)

Lemma 3: Let the functions f, g be nonnegative. Let $\eta > 0$. Denote by F_η the (finite) set $\{(i\eta, j\eta) \in R^2 : i, j \in N, (i-1)\eta \leq \sup f, (j-1)\eta \leq \sup g\}$. Suppose that there exists a neighborhood $\mathcal{U}$ of the point $(0, 0) \in R^2$ such that $(F_\eta + \mathcal{U}) \cap H = \emptyset$. Then η does not belong to the boundary of E.

Proof: The assumptions of Lemma 2 are fulfilled for some δ, ε satisfying the inequalities $\delta < \eta < \varepsilon$. (The numbers δ and ε can always be chosen such that for any point $P = (i\eta, j\eta) \in F_\eta$ we have $[i\varepsilon - (i+j)(\varepsilon - \delta), i\varepsilon) \times [j\varepsilon - (i+j)(\varepsilon-\delta), j\varepsilon) \subset P + \mathcal{U}$.)

Corollary 4: Let the functions f, g be nonnegative. Suppose that H contains no point $(u, v) \in R^2$ with u/v rational. Then $E = R_+$.

Proof: According to Lemma 1, the set E is nonvoid and by Lemma 3 it has no boundary points in R_+.

Lemma 5: Suppose that the ranges of f and g are nowhere dense in R and suppose that the range of g is countable. Then there is a point $(p, q) \in R^2$ with $p \leq \inf f$, $q \leq \inf g$ such that H is disjoint with every straight line which contains the point (p, q) and has a rational slope.

Proof: For all $u, v, r \in R$ we denote by $\ell_{u,v,r}$ the straight line in R^2 containing the point (u, v) and having the number r for its slope. The union $\bigcup_{u \in f(X)} \ell_{u,v,r}$ is a nowhere dense set. Further, the union $\mathcal{V} = \bigcup_{r \in Q} \bigcup_{v \in g(X)} \bigcup_{u \in f(X)} \ell_{u,v,r}$ is a set of the first category and therefore we can choose for the required point (p, q) any point in $(-\infty, \inf f] \times (-\infty, \inf g] - \mathcal{V}$.

<u>Theorem 6:</u> Let (X, L, s) be a QPS and let $f, g, f + g$ be bounded measurable functions on X. Let the ranges of f and g be nowhere dense sets in R and let the range of g be countable. Then $\int (f + g)\, ds = \int f\, ds + \int g\, ds$.

Proof: Suppose that $(p, q) \in R^2$ is a point with the properties of Lemma 5. Since by adding a constant function to $f, g, f + g$ the additivity remains unchanged, it is sufficient to prove the additivity for functions $f - p$, $g - p$. Thus, for the sake of simplicity, we can write f instead of $f - p$ and g instead of $q - p$ in the rest of the proof. According to Corollary 4, the sets K_ε belong to L and they also satisfy the condition (A) for all $\varepsilon \in R_+$. The condition (A) then gives

$$\int h_\varepsilon\, ds - \int f_\varepsilon\, ds - \int g_\varepsilon\, ds = \varepsilon \cdot s(K_\varepsilon) \in [0, \varepsilon].$$

So, for $\varepsilon \to 0_+$, the left-hand side converges to zero. Therefore $\int (f + g)\, ds - \int f\, ds - \int g\, ds = 0$. The proof is complete.

Let us remark in conclusion that there are examples showing that neither boundedness nor nowhere density can be omitted in the formulation of Theorem 6. (See [1], Example 6 and [4].) Also, the theorem is not valid for more than two functions even if they are finitely valued. (See [4] and [9].) We do not know however whether Theorem 6 remains valid when we relax the countability condition of g.

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