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CONSTRUCTION OF A WRINKLED FUNCTION

Wrinkled functions were introduced by Nina Bary in [1] and she called them "fonctions ridées". In our paper we construct another wrinkled function using an idea different from that of Nina Bary.

Let $a_i = 1/2^{i+1} + 3/4^{i+1}$ and let P be the perfect, nowhere dense subset of $[0,1]$, $P = \{x : x = \sum c_i a_i$ with c_i taking the values 0 and 1 only$\}$. Each point $x \in P$ is uniquely represented by $\sum_{i=1}^{\infty} c_i(x)a_i$. Let $V_n = \bigcup_{c_1} \cdots \bigcup_{c_n} \left[\sum_{i=1}^{n} c_i a_i, \sum_{i=1}^{n} c_i a_i + \sum_{i=n+1}^{\infty} a_i \right]$. Then $P = \bigcap_{n=1}^{\infty} V_n$ and $|V_n| = 2^n \sum_{i=n+1}^{\infty} a_i \longrightarrow 1/2$. Hence $|P| = 1/2$. For each $x \in P$ let $F_1(x) = \sum_{i=1}^{\infty} c_{2i}(x)2^i$ and $F_2(x) = \sum_{i=1}^{\infty} c_{2i-1}(x)/2^i$. Extending F_1 and F_2 linearly on each interval contiguous to P we have F_1 and F_2 defined and continuous on $[0,1]$. Define $F : [0,1] \longrightarrow [0,1] \times [0,1]$ by $F(x) = (F_1(x), F_2(x))$. Clearly $F(P) = [0,1] \times [0,1]$. For each irrational number $z \in (0,1)$ represented in base 2 by $\sum_{i=1}^{\infty} d_i(z)/2^i$ let $A_z = \{x \in P : F_1(x) = z\} = \{x \in P : c_{2i}(x) = d_i(z)\} = a_z + A$, where $a_z = \sum_{i=1}^{\infty} d_i(z)a_{2i}$ and $A = \{x \in P : c_{2i}(x) = 0, i = 1,2,\ldots\}$ and let $B_z = \{x \in P : F_2(x) = z\} = \{x \in P : c_{2i-1}(x) = d_i(z)\} = b_z + B$, where $b_z = \sum_{i=1}^{\infty} d_i(z)a_{2i-1}$ and $B = \{x \in P : c_{2i-1}(x) = 0, i = 1,2,\ldots\}$. Then A_z and B_z are perfect sets of measure zero.

Lemma 1. a) $F_2(A_Z) = [0,1]$ *and* $F_1(B_Z) = [0,1]$.

b) F_2 *is increasing on* A_Z *and* F_1 *is increasing on* B_Z.

Proof. We prove the lemma only for F_2 since for F_1 the proof is analogous. Part a) is evident. b) Let $x,y \in A_Z$, $x < y$. Then $x = a_Z + \sum_{i=1}^{\infty} c_{2i-1}(x)a_{2i-1}$ and $y = a_Z + \sum_{i=1}^{\infty} c_{2i-1}(y)a_{2i-1}$. Let n be a natural number such that $c_{2n-1}(x) = 0$, $c_{2n-1}(y) = 1$ and $c_{2i-1}(x) = c_{2i-1}(y)$ for each $i \leq n-1$. We have $F_2(x) \leq \sum_{i=1}^{n-1} c_{2i-1}(x)/2^i + \sum_{i=n+1}^{\infty} 1/2^i = \sum_{i=1}^{n-1} c_{2i-1}(x)/2^i + 1/2^n \leq F_2(y)$.

Lemma 2. *For each measurable subset* M *of* P *the set* $F(M)$ *is measurable and* $|F(M)| = 2|M|$.

Proof. Let $I_{c_1\ldots c_{2p}} = \left[\sum_{i=1}^{2p} c_i a_i, \sum_{i=1}^{2p} c_i a_i + \sum_{i=2p+1}^{\infty} a_i\right]$. Then $F(P \cap I_{c_1\ldots c_{2p}}) = \left[\sum_{i=1}^{p} c_{2i}/2^i, \sum_{i=1}^{p} c_{2i}/2^i + 1/2^p\right] \times \left[\sum_{i=1}^{p} c_{2i-1}/2^i, \sum_{i=1}^{p} c_{2i-1}/2^i + 1/2^p\right] = S_{c_1\ldots c_{2p}}$ and $\operatorname{int}(S_{c_1\ldots c_{2p}}) \cap \operatorname{int}(S_{c_1'\ldots c_{2p}'}) = \emptyset$ for $(c_1 \cdots c_{2p}) \neq (c_1' \cdots c_{2p}')$. We have

$$(1) \qquad |S_{c_1\ldots c_{2p}}| = 2|P \cap I_{c_1\ldots c_{2p}}| = 1/4^p .$$

Let $\varepsilon > 0$ and let M be a measurable subset of P. Let G and H be open sets such that $M \subset G$ and $F(M) \subset H$, $|F(M)| > |H| - \varepsilon$, $|M| > |G| - \varepsilon$. Then $G \cap F^{-1}(H)$ is an open neighborhood of M. Since $P = \bigcap_{p=1}^{\infty} [\bigcup_{c_1} \cdots \bigcup_{c_{2p}} (P \cap I_{c_1\ldots c_{2p}})]$ and $|I_{c_1\ldots c_{2p}}| \longrightarrow 0$, as $p \longrightarrow +\infty$, it follows that $G \cap F^{-1}(H) \cap P$ can be represented as a countable union of disjoint sets $P \cap I_{c_1\cdots c_{2p}}$, $p = 1,2,\ldots$. By (1) it follows that $2\cdot|G \cap F^{-1}(H) \cap P| = |F(G \cap F^{-1}(H) \cap P)|$. Since $M \subset G \cap F^{-1}(H) \cap P \subset G$ and $F(M) \subset F(G \cap F^{-1}(H) \cap P) \subset H$ and ε is arbitrary, we obtain $|F(M)| = 2|M|$. It follows that F satisfies Lusin's condition (N). By the continuity of F and the measurability of M, $F(M) = F(Q) \cup F(Z)$, where Q and $F(Q)$ are F_σ sets and Z and $F(Z)$ are null sets. Hence $F(M)$ is measurable.

<u>Definition [1]</u>. A continuous function F is a wrinkled function if for each set E of positive measure there exists a set E_1, $|E_1| = 0$, $E_1 \subset E$ such that F is monotone on E_1 and $|F(E_1)| > 0$.

<u>Lemma 3</u>. F_1 <u>and</u> F_2 <u>are wrinkled functions on</u> P.

<u>Proof</u>. It is known that if a set $E \subset \mathbb{R} \times \mathbb{R}$ is of positive measure, then by Fubini's Theorem there exist uncountably many lines parallel to a fixed straight line d such that the intersection of E with each of these parallel lines is of positive measure. By Lemma 2 $F(M)$ is of positive measure. Recall that $|A_Z| = 0$ for all z. Let $z \in (0,1)$ such that $|F_2(E_Z)| > 0$, z irrational, where $E_Z = \{x \in M : F_1(x) = z\}$. Because $E_Z \subset A_Z$, $|E_Z| = 0$, and by Lemma 1, F_2 is increasing on A_Z and on E_Z. For F_1 the proof is analogous.

<u>Lemma 4</u>. <u>Let</u> $f : [a,b] \longrightarrow \mathbb{R}$ <u>be a continuous function and</u> Q <u>a positive measurable subset of</u> $[a,b]$. <u>Let</u> g <u>be a differentiable function on</u> $[a,b]$. <u>If</u> f <u>is wrinkled on</u> Q, <u>then</u> $f + g$ <u>is wrinkled on</u> Q.

<u>Proof</u>. Let E be a positive measurable subset of Q and let $E_1 \subset E$, $|E_1| = 0$ such that $|f(E_1)| > 0$ and f is monotone on E_1. Suppose for example that f is increasing on E_1. Let $F(x) = f(x)$, $x \in \bar{E}_1$ and let F be linear on each interval contiguous to $\bar{E}_1$. Then F is increasing on $[a,b]$. Let $E_{+\infty} = \{x : F'(x) = +\infty\}$. By [3] page 270 we have $|E_{+\infty}| = 0$, $E_{+\infty} \subset \bar{E}_1$ and $|F(E_{+\infty})| > 0$. Let $G(x) = F(x) + g(x)$. Then $E_{+\infty} = \{x : G'(x) = +\infty\}$ and $g \in ACG_*$ on $[a,b]$. It follows that $G \in VBG_*$ on $[a,b]$. Suppose that $|G(E_{+\infty})| = 0$. By [3] (Theorem 7.2, page 230 and Theorem 4.6, page 271) it follows that $G \in (N)$ on $[a,b]$. Hence $G \in ACG_*$ on $[a,b]$. Thus F is AC on $[a,b]$, a contradiction. Since $|G(E_{+\infty})| > 0$, there exists a subset A of $E_{+\infty}$ such that G is increasing on A and $|G(A)| > 0$. (See [2], page 83.) Hence $f + g$ is wrinkled on Q.

We can glean the following from Lemma 3. For each interval I there is a nowhere dense perfect subset P of I and a continuous function g on I such that $|P| = (1/2)\cdot|I|$, g is wrinkled on P, g vanishes at the endpoints of I, and g is linear on each component interval of $I - P$. Make g vanish outside of I. Let $I_1 = [0,1]$. Let P_1 and g_1 be the set and function so associated with I_1 such that $\sup|g_1| = 1$. Let I_2 be an interval of $I_1 - P$ of maximal length. Associate P_2 and g_2 this way with I_2. Let I_3 be an interval of $I_1 - (P_1 \cup P_2)$ of maximal length. Associate P_3 and g_3 this way with I_3. In general, let I_{k+1} be an interval of $(I_1 - (P_1 \cup \cdots \cup P_k))$ of maximal length. Associate P_{k+1} and g_{k+1} with I_{k+1} this way. Then $|(\bigcup_{n=1}^{\infty} P_n)| = 1$, and $H = \sum_{n=1}^{\infty} (1/2^n)\cdot g_n$ is wrinkled on $[0,1]$. Indeed, let E be a measurable subset of $[0,1]$, $|E| > 0$. Since $|\cup P_i| = 1$, it follows that there exists some P_n such that $|E \cap P_n| > 0$. Since H is the sum of a linear function and $(1/2^n)\cdot g_n$ on P_n, by Lemma 4 it follows that H is a wrinkled function on $[0,1]$.

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References

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