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SOME CATEGORY BASES WHICH ARE EQUIVALENT TO TOPOLOGIES

1. Introduction.

In a series of papers (for example, [3-8] and many others) John Morgan has developed a theory of *category bases*, which unites many features of the topological theory of category, the theory of measure, and certain other classifications of point sets. (Below we will recall some of the fundamentals of the theory.) Many of the most familiar category bases are topological spaces, and one of the basic questions in the theory is: Which category bases are equivalent to topological spaces? In [6], Morgan showed that any category base in which every region contains a minimal region (so, in particular, any finite category base) is, via a natural construction, equivalent to a topological space. In an abstract presented to the AMS in January 1987, Z. Piotrowski and A. Szymanski announced some relevant (as of this writing,unpublished) results, two of which bear directly on our work here. First, they obtained (independently) the result in our example (i) of Section 3; roughly, any category base of power less than or equal to $\aleph_1$ is equivalent to a topology. Second, they announce the construction (under Martin's axiom) of a category base which is not equivalent to a topology. Thus, the answer to the basic question above is at least -- "Not all of them."

We now recall some definitions and fix some notation.

A pair $(X, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of a nonempty set X, is called a category base if the nonempty sets in $\mathcal{C}$, called regions, satisfy the following axioms :

1. $\bigcup\mathcal{C} = X$.
2. Let A be a region and $\mathcal{D}$ a nonempty family of disjoint regions which has power less than the power of $\mathcal{C}$. Then

   i. If $A \cap (\bigcup\mathcal{D})$ contains a region, then there is a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region.

   ii. If $A \cap (\bigcup\mathcal{D})$ contains no region, then there is a region $B \subset A$ which is disjoint from $\bigcup\mathcal{D}$.

It is easy to see that every topological space is a category base. Other examples of category bases are:

1. If $(X,\mathcal{B},\mu)$ is a σ-finite measure space, and $\mathcal{C}$ is the class of sets of positive μ-measure, then $(X,\mathcal{C})$ is a category base.
2. (Assume the continuum hypothesis.) If μ is a Hausdorff measure on $\mathbf{R}^n$ (n-dimensional Euclidean space), and $\mathcal{C}$ is the class of closed subsets of positive μ-measure, then $(\mathbf{R}^n,\mathcal{C})$ is a category base.
3. If $\mathcal{C}$ is the class of non-empty perfect subsets of $\mathbf{R}$, then $(\mathbf{R},\mathcal{C})$ is a category base.

It has been known for a long time that the category base of example 1 is equivalent to a topology (see [10]). We shall show presently that the category bases of examples 2 and 3 enjoy the same property.

A set $S \subset X$ is *singular* if for all regions C there exists a region $C' \subset C$ such that $S \cap C' = \varnothing$. A set $M \subset X$ is *meager* if M is a countable union of singular sets. The class of meager sets will be denoted $\mathcal{M}(\mathcal{C})$. A set $A \subset X$ has the *Baire property* if, for all regions C, there exists a region $C' \subset C$ such that ($C' \cap A$ is meager or $C' - A$ is meager). The class of sets with the Baire property will be denoted $\mathcal{B}(\mathcal{C})$.

Let $(X,\mathcal{C})$ and $(X,\mathcal{D})$ be category bases. $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if $\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{D})$ and $\mathcal{B}(\mathcal{C}) = \mathcal{B}(\mathcal{D})$. For any category base $(X,\mathcal{C})$, $(X,\mathcal{C}-\{\varnothing\})$ is a category base which is equivalent to $(X,\mathcal{C})$. We shall therefore assume, without loss of generality, that every set in $(X,\mathcal{C})$ is a region (i.e., is non-empty).

2. Main results.

Throughout this section, fix a category base $(X,\mathcal{C})$. Call a set $\mathcal{D}$ of non-empty subsets of X a *basis* if $X = \bigcup\mathcal{D}$ and $\mathcal{D} \cup \{\varnothing\}$ is closed under finite intersection. It is easy to check that, if $\mathcal{D}$ is a basis, then $\mathcal{D}$ is equivalent to $\mathcal{T}$, the topology generated by $\mathcal{D}$ (i.e., $\mathcal{T}$ consists of all unions of arbitrary subsets of $\mathcal{D}$). We shall therefore use topologies and bases interchangeably.

Lemma 1: Let $\mathcal{D}$ be a basis such that $\mathcal{D} \subset \mathcal{B}(\mathcal{C}) - \mathcal{M}(\mathcal{C})$, and such that for all $C \in \mathcal{C}$ there exists $D \in \mathcal{D}$ such that $D \subset C$. Then $\mathcal{D}^* = \{ D - M : D \in \mathcal{D},\ M \in \mathcal{M}(\mathcal{C}) \}$ is a basis which is equivalent to $\mathcal{C}$.

Proof: We must show that $\mathcal{M}(\mathcal{D}^*) = \mathcal{M}(\mathcal{C})$ and $\mathcal{B}(\mathcal{D}^*) = \mathcal{B}(\mathcal{C})$.

i) $\mathcal{M}(\mathcal{C}) \subset \mathcal{M}(\mathcal{D}^*)$: Let $M \in \mathcal{M}(\mathcal{C})$. Then for all $D \in \mathcal{D}^*$, $D - M \in \mathcal{D}^*$; thus M is $\mathcal{D}^*$-nowhere dense, and, a fortiori, $M \in \mathcal{M}(\mathcal{D}^*)$.

ii) $\mathcal{M}(\mathcal{D}^*) \subset \mathcal{M}(\mathcal{C})$: It is enough to show that every $\mathcal{D}^*$-nowhere dense set is $\mathcal{C}$-meager, so suppose M is $\mathcal{D}^*$-nowhere dense. By the Banach Category Theorem (see, e.g., [5], p. 23) if $M \notin \mathcal{M}(\mathcal{C})$, then there exists $C \in \mathcal{C}$ such that for all $C' \in \mathcal{C}$ such that $C' \subset C$, $M \cap C' \notin \mathcal{M}(\mathcal{C})$. Therefore let $C \in \mathcal{C}$; we will find $C' \in \mathcal{C}$ such that $C' \subset C$ and $C' \cap M \in \mathcal{M}(\mathcal{C})$. Indeed, by hypothesis there exists $D \in \mathcal{D}$ such that $D \subset C$, and $D' \in \mathcal{D}^*$ such that $D' \subset D$ and $D' \cap M = \varnothing$. Since $D' \in \mathcal{B}(\mathcal{C}) - \mathcal{M}(\mathcal{C})$ and $D' \subset C$, there exists $C' \subset C$ such that $C' - D' \in \mathcal{M}(\mathcal{C})$. But $C' \cap M \subset C' - D'$, so $C' \cap M \in \mathcal{M}(\mathcal{C})$.

iii) $\mathcal{B}(\mathcal{C}) \subset \mathcal{B}(\mathcal{D}^*)$: Suppose $A \in \mathcal{B}(\mathcal{C})$. Given an arbitrary $D \in \mathcal{D}^*$, there exists $C \in \mathcal{C}$ such that $C - D \in \mathcal{M}(\mathcal{C})$. By assumption, there exists $C' \in \mathcal{C}$ such that $C' \subset C$ and ($C' \cap A \in \mathcal{M}(\mathcal{C})$ or $C' - A \in \mathcal{M}(\mathcal{C})$). There exists $D' \in \mathcal{D}$ such that $D' \subset C'$. Now $D' - D \subset C - D \in \mathcal{M}(\mathcal{C})$, so $D' \cap D = D' - (D' - D) \in \mathcal{D}^*$, and ($(D' \cap D) \cap A \in \mathcal{M}(\mathcal{C})$ or $(D' \cap D) - A \in \mathcal{M}(\mathcal{C})$). Since $\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{D}^*)$, by definition we have $A \in \mathcal{B}(\mathcal{D}^*)$.

iv) $\mathcal{B}(\mathcal{D}^*) \subset \mathcal{B}(\mathcal{C})$: Let $A \in \mathcal{B}(\mathcal{D}^*)$. Given an arbitrary $C \in \mathcal{C}$, there exists $D \in \mathcal{D}$ such that $D \subset C$, and $D' \in \mathcal{D}^*$ such that $D' \subset D$ and ($D' \cap A \in \mathcal{M}(\mathcal{D}^*)$ or $D' - A \in \mathcal{M}(\mathcal{D}^*)$). There exists $C' \in \mathcal{C}$ such that $C' \subset C$ and $C' - D' \in \mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{D}^*)$, so since

$C' \cap A \subset (D' \cap A) \cup (C' - D')$ and $C' - A \subset (D' - A) \cup (C' - D')$, we have ($C' \cap A \in \mathcal{M}(\mathcal{C})$ or $C' - A \in \mathcal{M}(\mathcal{C})$). Thus $A \in \mathcal{B}(\mathcal{C})$, which completes the proof of the lemma.

The main difficulty in turning a category base $\mathcal{C}$ into an equivalent topology is what to do when you come to $C_1, \dots, C_n \in \mathcal{C}$ such that $C_1 \cap \dots \cap C_n$ is non-empty but meager. Indeed, if this never happens the problem is easy.

Proposition 2: If, for all $C_1, \dots, C_n \in \mathcal{C}$, $C_1 \cap \dots \cap C_n \in \mathcal{M}(\mathcal{C})$ only if $C_1 \cap \dots \cap C_n = \varnothing$, then

$$\{ C_1 \cap \dots \cap C_n - M : C_1, \dots, C_n \in \mathcal{C},\ M \in \mathcal{M}(\mathcal{C}) \}$$

is a basis which is equivalent to $\mathcal{C}$.

This proposition follows immediately from the preceding lemma. Unfortunately, the hypothesis of Proposition 2 is rarely satisfied unless $\mathcal{C}$ is a topology to begin with. To get a somewhat more useful result, define

$$\mathcal{N}(\mathcal{C}) = \{ C_1 \cap \dots \cap C_n : C_1, \dots, C_n \in \mathcal{C} \text{ and } C_1 \cap \dots \cap C_n \in \mathcal{M}(\mathcal{C}) \}.$$

Theorem 3: Assume that $\mathcal{C}$ is infinite, and that every region in $\mathcal{C}$ is abundant. Suppose further that for all $\mathcal{E} \subset \mathcal{N}(\mathcal{C})$ such that the power of $\mathcal{E}$ is less than the power of $\mathcal{C}$, $\bigcup \mathcal{E} \in \mathcal{M}(\mathcal{C})$. Then $\mathcal{C}$ is equivalent to a topology.

Proof: Enumerate $\mathcal{C} = \{ C_\xi : \xi < \kappa \}$ (so $\mathcal{C}$ has κ elements). For all $\xi < \kappa$, let

$$E_\xi = \bigcup \{ C_\xi \cap C_{\eta_1} \cap \dots \cap C_{\eta_n} : \eta_1, \dots, \eta_n < \xi \text{ and } C_\xi \cap C_{\eta_1} \cap \dots \cap C_{\eta_n} \in \mathcal{M}(\mathcal{C}) \}$$

and $D_\xi = C_\xi - E_\xi$. Let

$$\mathcal{D} = \{ D_{\xi_1} \cap \dots \cap D_{\xi_n} : \xi_1, \dots, \xi_n < \kappa \} \cup \{X\} - \{\varnothing\}.$$

We finish the proof by showing that $\mathcal{D}$ satisfies the hypotheses of Lemma 1. Indeed $\mathcal{D}$ is clearly a basis, and for $\xi < \kappa$, $D_\xi \subset C_\xi$. Now consider a typical element

$D_{\xi_1} \cap \dots \cap D_{\xi_n}$ of $\mathcal{D}$, and say $\xi_1 > \xi_2, \dots, \xi_n$. Of course,

$$D_{\xi_1} \cap \dots \cap D_{\xi_n} = (C_{\xi_1} \cap \dots \cap C_{\xi_n}) - (E_{\xi_1} \cup \dots \cup E_{\xi_n}).$$

Each E_ξ is a union of fewer than κ elements of $\mathcal{N}(\mathcal{C})$, so by hypothesis

$E_{\xi_1} \cup \dots \cup E_{\xi_n} \in \mathcal{M}(\mathcal{C})$. If $C_{\xi_1} \cap \dots \cap C_{\xi_n} \in \mathcal{M}(\mathcal{C})$, then by construction $C_{\xi_1} \cap \dots \cap C_{\xi_n} \subset E_{\xi_1}$, so $D_{\xi_1} \cap ... \cap D_{\xi_n} = \varnothing$, contrary to the hypothesis that this set is not in $\mathfrak{D}$. Therefore

$$C_{\xi_1} \cap ... \cap C_{\xi_n} \in \mathfrak{B}(\mathfrak{C}) - \mathfrak{M}(\mathfrak{C}) \text{ and so } D_{\xi_1} \cap ... \cap D_{\xi_n} \in \mathfrak{B}(\mathfrak{C}) - \mathfrak{M}(\mathfrak{C}).$$

We have verified that $\mathfrak{D}$ satisfies the hypotheses of Lemma l. The proof of Theorem 3 is complete.

3. Examples.

i) If $\mathfrak{C}$ has power less than or equal to $\aleph_1$ (the first uncountable cardinal number) and every region in $\mathfrak{C}$ is abundant, then Theorem 3 applies. (The union in its hypothesis must be a countable union).

In particular, assume the continuum hypothesis, let μ be a Hausdorff measure on $\mathbf{R}^n$, and let $\mathfrak{C}$ be the class of closed subsets of positive μ-measure. In [5], it is shown that $(\mathbf{R}^n, \mathfrak{C})$ is a category base, $\mathfrak{B}(\mathfrak{C})$ is the class of μ-measurable sets, and $\mathfrak{M}(\mathfrak{C})$ is the class of sets having no subset of finite, positive measure. Since under the continuum hypothesis $\mathbf{R}^n$ has $\aleph_1$ closed subsets, we have, by Theorem 3:

There exists a topology $\mathfrak{T}$ on $\mathbf{R}^n$ such that "$\mathfrak{T}$-Baire property = μ–measurable."

ii) The Marczewski sets (see [1], [2], [4] and [11]).

In [11], Marczewski investigated a class of sets which became known as the Marczewski sets. A set A of real numbers is a *Marczewski set* if every perfect set of real numbers P has a perfect subset P' such that either $P' \subset A$ or $A \cap P' = \varnothing$. A set A is *Marczewski singular* if every perfect set P has a perfect subset P' such that $A \cap P' = \varnothing$.

It is shown in [4] that $(\mathbf{R}, \mathfrak{P})$ is a category base, where $\mathfrak{P}$ is the class of perfect subsets of $\mathbf{R}$. Furthermore, $\mathfrak{B}(\mathfrak{P})$ is the class of Marczewski sets, and $\mathfrak{M}(\mathfrak{P})$ is the class of Marczewski-singular sets.

As we shall now show, Theorem 3 applies to $(\mathbf{R}, \mathfrak{P})$. Indeed, let P be a perfect set, and let $\mathfrak{E}$ be a subset of $\mathfrak{N}(\mathfrak{P})$ of power less than the continuum. Then every $E \in \mathfrak{E}$ is closed and has no perfect subset; by the Cantor-Bendixson theorem each such E is countable. Therefore $\cup\mathfrak{E}$ has power less than the continuum. By a well-known theorem, P has continuum many disjoint perfect subsets. One such subset must be disjoint from $\cup\mathfrak{E}$. Thus $\cup\mathfrak{E}$ is Marczewski singular and so $\mathfrak{P}$-meager, and so $(\mathbf{R}, \mathfrak{P})$ satisfies the hypotheses of Theorem 3.

Therefore we have : There exists a topology $\mathfrak{T}$ on $\mathbf{R}$ such that $\mathfrak{T}$-Baire property = Marczewski set, and $\mathfrak{T}$-meager = Marczewski singular.

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