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A SYMMETRIC DENSITY PROPERTY FOR MEASURABLE SETS

In [2] we established the following result:

THEOREM A. Let W and B be open subsets of a real interval whose union has full measure. If for each x, the set $\{h>0 \mid x-h \in W \text{ and } x+h \in B\}$ has density zero at zero then these sets are all empty.

This was then used to prove the following :

COROLLARY B. If a continuous real function has a non-negative lower approximate symmetric derivative on some interval then it is non-decreasing on that interval.

In this note we show how to extend Theorem A to get:

THEOREM 1. Let W and B be measurable subsets of a real interval whose union has full measure. If for each x, the set $\{h>0 \mid x-h \in W \text{ and } x+h \in B\}$ has density zero at zero then these sets all have measure zero.

This is an equivalent form of Query 184 in Volume 12, no. 2 of the Exchange, which also appeared as conjecture C(6) in Foran and Larson [3]. It settles all six conjectures in that paper.

We will also prove:

THEOREM 2. If a measurable real function has a non-negative lower approximate symmetric derivative then it is non-decreasing on the set of points for which it is approximately continuous.

This settles a conjecture in Larson [4].

Thomson, [5], established partitioning properties for covers of an interval that are related to the ordinary, approximate, and symmetric derivatives. Theorem 3 proves a corresponding property for the approximate symmetric case. The appropriate definitions will be included under the heading "Notation".

THEOREM 3. If S is an approximate symmetric cover of an interval then S partitions almost every concentric subinterval.

Our goal here, of course, is to establish the truth of Theorems 1-3. Theorem A (proved in [2]) will be our main technical tool. In particular we will show Theorem A $\rightarrow$ Theorem 1, and Theorem 1 $\rightarrow$ both Theorem 2 and Theorem 3. In fact, what we are establishing in this paper is the equivalence of all four theorems, in the sense that without knowledge of a proof of any of them, each of them implies the others. This is because it can be easily demonstrated that Theorem 3 $\rightarrow$ Theorem 2 by techniques in Thomson [5], Theorem 2 $\rightarrow$ Theorem 1 by considering the characteristic function of the set W, and Theorem 1 reduces to Theorem A in the case where W and B are open.

NOTATION: Let A and B be subsets of the real line and x a real number. Then let $\lambda(A)$ ($\lambda_*(A)$, $\lambda^*(A)$) denote the (inner, outer) Lebesgue measure of A. Let cl(A) denote the closure of A. Let AB(x) denote $\{h>0 \mid x-h\in A \text{ and } x+h\in B\}$. Let $A\subset_d B$ denote the relationship $A\subset B$ and every point of A is a density point of B. If g is a real function then A(g) denotes the set of approximate continuity points of g. For an interval (a,b), a collection ,S ,of subintervals is an approximate symmetric cover of (a,b) if, for each x, there is a measurable set H_x, with density 1 at 0, so that $\{[x-h,x+h] \mid h\in H_x\}\subset S$. We say S partitions the subinterval [c,d] if there are numbers $c=x_0<x_1<\ldots<x_n=d$ so that $[x_{i-1},x_i]\in S$ for $1\le i\le n$.

__PROOF OF THEOREM 1__: Let W and B be measurable subsets of a real interval whose union has full measure and such that for each x, WB(x) has density zero at zero. If W and B intersect in positive measure then they can't satisfy this density condition, so we may assume without loss of generality that W and B are disjoint. Let $W'\subset W$ and $B'\subset B$ be the density points of W and B respectively. Then W' and B' are also disjoint, and by the Density Theorem, (see [6]), $W'\cup B'$ has full measure. For convenience we will assume our interval is [0,1]. Let ϵ be any real number in (0,1/2). Let W_1 and B_1 be closed subsets of W' and B' respectively with $\lambda(W_1\cup B_1)>1/2$. Let S_{1W} be the collection of open intervals, N, whose endpoints are in W' and such that for any subinterval N' which shares at least one endpoint with N, we have $\lambda(N'\cap W')/\lambda(N') > 1-\epsilon$.

We wish to show that S_{1W} covers W_1 in the sense of Vitali. (see [6] pp. 105-109). By a theorem of Luzin and Menchoff (see [1] pg. 27), we may choose a perfect set P such that $W_1\subset_d P \subset_d W'$. Let x be in W_1 and let δ be any positive real number. By choice of P there is a symmetric interval, (a,b), about x of size less than δ such that the relative measure of P in this interval is at least $1-\epsilon/3$. Let a' be the maxmimum c in $(a,b]$ such that $\lambda(P\cap(a,c))/(c-a) \le 1-\epsilon$ and let $a'=a$ if no such c exists. Let b' be the minimum c in $[a,b)$ such that $\lambda(P\cap(c,b))/(b-c) \le 1-\epsilon$, and $b'=b$ if no such c exists. Clearly a' is less than x and b' is greater than x since otherwise $\lambda(P\cap(a,b))/(b-a) \le 1-\epsilon/2$. Hence we need only show that a' and b' are in W'. Since for any c in (a',b) we have $\lambda(P\cap(a',c))/(c-a') > 1-\epsilon$, and since P is closed, a' must be in P. Similarly, b' is in P, and hence (a',b') is in S_{1W} as required.

Let S_{1B} be the similar collection for B' which covers B_1 in the sense of Vitali, and let $S_1 = S_{1W} \cup S_{1B}$. Then by Vitali's Lemma, there exists a finite collection, C_1, of disjoint open intervals in S_1 such that $\lambda(\cup C_1) > 1/2$. Let σ_1 be the size of the smallest interval in C_1.

Let W_2 and B_2 be closed subsets of $W'-\mathrm{cl}(\cup C_1)$ and $B'-\mathrm{cl}(\cup C_1)$ respectively such that $\lambda(W_2 \cup B_2 \cup \cup C_1) > 3/4$. Let S_{2W} be the collection of open intervals, $N \subset (0,1) - \mathrm{cl}(\cup C_1)$, of length less than σ_1, whose endpoints are in W', and such that for any subinterval N' which has at least one endpoint in common with N, we have $\lambda(N'\cap W')/\lambda(N) > 1-\epsilon/2$. Let S_{2B} be the corresponding collection for B, and $S_2 = S_{2W} \cup S_{2B}$. Then S_2 covers $W_2 \cup B_2$ in the sense of Vitali. Hence by Vitali's Lemma, there exists a finite collection, C_2 of disjoint open intervals in S_2 such that $\lambda(\cup C_1 \cup \cup C_2) > 3/4$. Let σ_2 be the size of the smallest interval in C_2.

Continuing in this manner we obtain a countable collection, $C = \cup C_i$, of disjoint open intervals with $\lambda(\cup C) = 1$, such that for each interval N in C

with measure less than σ_{i-1} (let σ_0 be any number > 1) and each subinterval N' which shares at least one endpoint with N, either both endpoints of N are in W' and $\lambda(N'\cap W')/\lambda(N') > 1-\epsilon/i$, or both endpoints of N are in B' and $\lambda(N'\cap B')/\lambda(N') > 1-\epsilon/i$.

Let W" be the union of neighborhoods in C whose endpoints are in W' and let B" be the union of neighborhoods whose endpoints are in B'. Every neighborhood in C is in either W" or B" and since W' and B' are disjoint, no neighborhood is in both. Hence W" and B" are disjoint open sets whose union has full measure.

In order to apply Theorem A. we need to show that for each x, W"B"(x) has density zero at zero. If x is a member of $\cup C$ then this follows since W" and B" are disjoint. If x is in an endpoint of some component of $\cup C$ then this follows since either x is an endpoint of a component of W" and also a density point of W or else x is an endpoint of a component of B" and also a density point of B. It cannot be both since W' and B' are disjoint. Since $\epsilon<1/2$ each component of W" is mostly in W and each component of B" is mostly in B. Therefore, x must either be a density point of W" or a density point of B" which implies that W"B"(x) has density zero at zero as required. For other values of x, (ie. x is not in the closure of any component of $\cup C$) let i be any large integer and let N(x) be any symmetric neighborhood of x which intersects only those neighborhoods in C with length less than σ_{i-1}, and let $N_x=\{h>0|x+h \in N(x)\}$, the translate of N(x). Then $\lambda(W''\cap N(x)-W)/\lambda(N(x)) < \epsilon/i$ and $\lambda(B''\cap N(x)-B)/\lambda(N(x)) < \epsilon/i$. Hence $\lambda(W''B''(x) \cap N_x) < \lambda(WB(x) \cap N_x) + 2\epsilon\lambda(N(x))/i$. Since i was arbitrary and since WB(x) has density zero at zero, it follows that W"B"(x) has density zero at zero as required.

By Theorem A, it now follows that $W''B''(x)$ is empty for each x. In other words there is no element of B" which is to the right of an element of W". Hence for each x, we have

$$\begin{aligned}\lambda(WB(x)) &= \lambda(W\cap B''\ B\cap B''(x)) + \lambda(W\cap B''\ B\cap W''(x)) + \lambda(W\cap W''\ B\cap W''(x))\\ &< \lambda(W\cap B'') + \lambda(W\cap B'') + \lambda(B\cap W'')\\ &< 3\epsilon.\end{aligned}$$

Since ϵ is arbitrary it follows that the measure of $WB(x)$ is zero. □

PROOF OF THEOREM 2: Let f be a measurable real function whose lower approximate symmetric derivative is nonnegative. For $\epsilon>0$ let $g(x)=f(x)+\epsilon x$. Then $A(g)=A(f)$. Suppose $c<d$ are two points in $A(g)$ and $g(c)>g(d)$. Pick $y\in(g(d),g(c))$. Then since the lower approximate symmetric derivative of g is positive, $W=g^{-1}[y,\infty)$ and $B=g^{-1}(-\infty,y]$ clearly satisfy the hypothesis of Theorem 2. Let $m=(c+d)/2$. Since W has density one at c and B has density one at d, $\lambda(WB(m))>0$, contradicting Theorem 1. Thus $g|_{A(g)}$ is non-decreasing. Since ϵ is arbitrary $f|_{A(f)}$ is non-decreasing. □

THEOREM 3. Let S be an appoximate symmetric cover of (a,b) and c the midpoint of (a,b). For almost every $0<h<(b-a)/2$, $[c-h,c+h]$ is partitioned by S.

PROOF: We may assume that $c=0$. Let $W'=\{h>0\,|\,[-h,h] \text{ is partitioned by } S\}$, and $B'=(0,b)-W'$. Let W be an F_σ set contained in W' with $\lambda(W)=\lambda_*(W')$. Note that $\lambda(W)>0$ since $H_0\subset W'$. Let $B=(0,b)-W$. Then $\lambda(B)=\lambda^*(B')$. We show that $WB(x)$ has density 0 at 0 for all x in (0,b). Suppose $x\in(0,b)$ and $WB(x)$ has positive upper density at 0. Then, for $y=-x$, $WB(x)\cap H_x\cap H_y$ also has positive upper density at 0. This says, for some $\delta>0$ and for arbitrarily small neighborhoods, N, of 0, we have $\lambda(WB(x)\cap H_x\cap H_y\cap N)>\delta\lambda(N)$. This gives $\lambda^*(W'B'(x)\cap H_x\cap H_y\cap N)>\delta\lambda(N)$ since $\lambda(B)=\lambda^*(B')$ and $W\subset W'$. Since $h\in W'B'(x)$ means $[-(x-h),x-h]$ is partitioned by S but $[x-h,x+h]$ or $[y-h,y+h]$ is not, the last inequality contradicts the fact that H_x and H_y have density 1 at 0. Thus $WB(x)$ has density 0 at 0. By Theorem 1, $WB(x)$ has measure 0 for all x in $(0,b)$. Recall that W has positive measure in every neighborhood of 0, so B cannot have any subsets of positive measure to the right of 0. Thus $\lambda(B)=0$ which says $\lambda(B')=0$. □

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