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H. Pawlak, R. Pawlak, Institute of Mathematics, Łódź University, Łódź, Poland.

FUNDAMENTAL RINGS FOR CLASSES OF DARBOUX FUNCTIONS

Many interesting results are connected with addition and multiplication of Darboux functions, see, for example: [2],[3],[5],[7], [10],[11]. We have investigated some problems connected with the possibility of constructing some rings of Darboux functions and their properties ([8],[9]). In this paper we try to answer the following question: for which classes of Darboux functions can we construct a fundamental ring. Our paper does not answer this question completely leaving the general case is open (Problems 1,2). Our considerations are similar, in a way, to the research on the simultaneously maximal additive and multiplicative classes for some family of Darboux function ([1],[2],[5]).

We now give the main definitions and notation. The symbol R denotes the set of real numbers with the natural topology of the line; in a case of another topology we specifically state this fact. Let $\overline{A}$ denote the closure of A (in the natural topology) and $\mathcal{L}(A)$, the set of all components of A also in the natural topology.

Throughout this paper we consider real functions. For a family H of functions, the symbol $H_{|A}$ denotes the set of all restrictions $h_{|A}$, where $h \in H$. We consider the set $C_H = \bigcap_{h\in H} C_h$, where C_h denotes the set of the continuity points of h. Then $D_H = R \setminus C_H$. If $H = \{h\}$, we write D_h.

The symbol C denotes the class of continuous functions (in the nat. top.), and $C(T)$, the family of the continuous function $f : (R,T) \to R$, where T is some topology in R. If $\mathcal{T}$ is a family of topologies on the real line, then $C(\mathcal{T}) = \{C(T): T \in \mathcal{T}\}$.

If T is some topology on R, then by a T-open set we mean a set open with respect to T, and by a T-continuous function, a continuous function $f : (R,T) \to R$. The terms an open set, a closed set, a continuous function will refer to the natural topology.

We say that a topology T is connected if the space (R,T) is connected.

We say that two families of Darboux functions H_1 and H_2 are compatible (with respect to conditions $(*)$) if there exists a ring $\mathbb{P}$ of Darboux functions (fulfiling conditions $(*)$) such that $H_1, H_2 \subset \mathbb{P}$.

Let $\mathbb{P}$ be some ring of functions and let H be some family of functions. Then $\mathbb{P}(H)$ denotes the ring consisting of all functions of the form

$$g = g_0 + g_1 \cdot h_{1,1}^{s_{1,1}} \cdot h_{1,2}^{s_{1,2}} \cdots h_{1,k_1}^{s_{1,k_1}} + \cdots + g_t \cdot h_{t,1}^{s_{t,1}} \cdot h_{t,2}^{s_{t,2}} \cdots$$
$$\cdots h_{t,k_t}^{s_{t,k_t}}$$

where $g_i \in \mathbb{P}$, $h_{j,p} \in H$ and $s_{j,p}$ are natural numbers.

If H is a class of Darboux functions, then we say that a ring $\mathbb{P}$ of real functions is fundamental for H, if $\mathbb{P}(H)$ is a ring of Darboux functions. We say that a class P of fundamental rings for H is complete (with respect to conditions (*)) if for every family of functions H_1 compatible with H (resp. to (*)) there exists a ring $\mathbb{P} \in P$ which is fundamental for $H \cup H_1$.

We say that a real number α is a right-side cluster number of f at x_o relative to a set A if there exists a sequence $\{x_n\} \subset A$ such that $x_n \searrow x_o$ and $\lim_{n\to\infty} f(x_n) = \alpha$. Similarly we define left-side cluster numbers. The set of all right-side (left-side) cluster numbers of f at x_o relative to a set A we denote by $L_A^+(f,x_o)$ $(L_A^-(f,x_o))$. If $A = R$, we just write $L^+(f,x_o)$ $(L^-(f,x_o))$.

Let H be some class of functions, $\eta > 0$ and $A \subset R$. Then $\overline{H}_\eta^+(A,x_o) = \{h \in H: \exists(\beta \in L_A^+(h,x_o))\ \beta - h(x_o) > \eta\}$, $\underline{H}_\eta^+(A,x_o) = \{h \in H: \exists(\beta \in L_A^+(h,x_o))\ h(x_o) - \beta > \eta\}$. Similarly we define $\overline{H}_\eta^-(A,x_o)$ and $\underline{H}_\eta^-(A,x_o)$.

We say that a net $\{x_\sigma\}_{\sigma\in\Sigma} \subset R$ is decreasing if

$$\forall(\sigma_1,\sigma_2\in\Sigma)\quad \sigma_1 \rightarrow \sigma_2 \Leftrightarrow x_{\sigma_1} \geq x_{\sigma_2},$$

where $\rightarrow$ is a directing relation in Σ. Analogously we define an increasing net.

The aim of this paper is to find conditions on a family H of Darboux functions, under which there exists a fundamental ring for H. We assume the following definitions.

DEFINITION 1. We say that a family H possesses the <u>property of Young</u> if for every x there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \nearrow x \swarrow y_n$ and $\lim_{n\to\infty} f(x_n) = f(x) = \lim_{n\to\infty} f(y_n)$, for every $f \in H$.

If H possessses the property of Young (consists of functions of first class of Baire), then we write $H \subset Y (H \subset B_1)$. Of course if $H = \{f\}$, the property of Young for a family $\{f\}$ is identical to the well known property of Young for f. (See [1],[2]). The following proposition is easily proves. (See [8] and [9]).

PROPOSITION 1. Let f_i $(i=1,\dots,k)$ be Darboux functions such that $D_{f_i} = \{x_o\}$ $(i=1,\dots,k)$. Then the functions f_i $(i=1,\dots,k)$ belong to a common ring of Darboux functions which includes the constant functions if and only if the family $\{f_i\colon\ i = 1,\dots,k\}$ possesses the property of Young.

DEFINITION 2. Let H be a class of real functions defined on a set $A \subset \mathbb{R}$. We say that x_o is <u>a right sided upper Darboux point for H, with respect to A</u> (written $x_o \in \mathrm{Db}\dot{x}^{+}(H,A)$, if x_o is an isolated point from the right side in A or for every $\eta > 0$ there exists $\delta_o > 0$ such that for every $\delta \le \delta_o$, $C \in \mathcal{L}(A \cap (x_o, x_o+\delta))$ and $h \in \overline{H^{+}_{\eta}}(A, x_o)$ the following inequalities hold

$$(*)\qquad h^{-1}(h(x_o) + \eta) \cap A \cap (x_o, x_o+\delta) \neq \emptyset \neq$$

$$\neq \bigcap_{h\in H} h^{-1}((h(x_o) - \eta, h(x_o) + \eta)) \cap C.$$

Analogously we define a right sided lower Darboux point, left-sided upper and left sided lower Darboux point by replacing $\overline{H}^{+}_{\eta}(A,x_o)$ with $\underline{H}^{+}_{\eta}(A,x_o)$, $\overline{H}^{-}_{\eta}(A,x_o)$ and $\underline{H}^{-}_{\eta}(A,x_o)$ respectively.

The notion of Darboux point and one sided Darboux point are defined in the obvious way. If $H = \{h\}$, we say refer to a right sided upper (right sided lower,...) Darboux point for h.

PROPOSITION 2. Let $f : R \to R$. Then x_o is a Darboux point from the right (left) side for f ([6]) if and only if x_o is a right (left) sided upper and lower Darboux point for f (with respect to R).

Before we formulate and prove the fundamental theorem of this paper, we give the following definition.

DEFINITION 3. We say that a family H fulfils the condition $(\mathbb{D})$ if there exists a nonempty set $A \subset C_H$ such that:

1. every element $x \in R$ is a Darboux point for H with respect to A
2. $H_{|\bar{C}} \subset B_1$ for every $C \in \ell(R \setminus \bar{A})$.

If additionaly $H_{|\bar{C}} \subset Y$ for every $C \in \ell(R \setminus \bar{A})$, then we say that H fulfils condition $(\mathbb{D}Y)$.

THEOREM 3. Let H be a family of the functions fulfiling condition $(\mathbb{D}Y)$. Then H is a family of Darboux functions and there exists a nonempty class $\mathcal{T}$ of connected topologies finer than the natural topology of the line such that $C(\mathcal{T})$ is the complete class of fundamental rings for H, with respect to condition $(\mathbb{D})$.

P r o o f. Let $\mathfrak{a}$ be a family of all sets $A \subset C_H$ such that H fulfils the condition $(\mathbb{D}Y)$ with respect to A. Let $A \in \mathfrak{a}$. We describe now a class of topologies. First we construct one topology in this class. Let $B = R \setminus \bar{A}$. For every $t \in \bar{C}$, where $C \in \ell(B)$ let $\{t_\sigma^+\}_{\sigma\in\Sigma}$ and $\{t_\delta^-\}_{\delta\in\Delta}$ be decreasing and increasing nets converging to t and such that for every finite set of functions $h_1, \dots, h_k \in H$ there exist subnets $\{t_{\sigma'}^{'+}\}_{\sigma'\in\Sigma'}$ of $\{t_\sigma^+\}_{\sigma\in\Sigma}$ and $\{t_{\delta'}^{'-}\}_{\delta'\in\Delta'}$ of $\{t_\delta^-\}_{\delta\in\Delta}$ fulfiling the following condition:

$$(1) \qquad \lim_{\sigma'\in\Sigma'} h_i(t_{\sigma'}^{'+}) = h_i(t) = \lim_{\delta'\in\Delta'} h_i(t_{\delta'}^{'-}) \quad \text{for } i=1,\dots,k.$$

Since $H_{|\bar{C}} \subset Y$, the family of nets fulfiling above condition is nonempty. (Of course if t is a left endpoint of the closure of a component C, then we consider only $\{t_\sigma^{'}\}_{\sigma\in\Sigma}$ and analogously in the case a right endpoint of $\bar{C}$). For every $C \in \ell(B)$ let T_C denote a topology in $\bar{C}$ finer than the natural topology of the segment $\bar{C}$ such that every $U \in T_C$ is a F_σ set (in the natural topology of the line) and for every $U \in T_C$ and $t \in U$ there exist $\sigma_o \in \Sigma$ and $\delta_o \in \Delta$ such that $t_\sigma^+ \in U$ and $t_\sigma^- \in U$ for every $\sigma \geq \sigma_o$ and $\delta \geq \delta_o$. (Of course, at least one such topology exists).

For $t \in \bar{C}$ we put $B_{T_C}(t) = \{U \in T_C : t \in U \wedge U \neq \bar{C}\}$. For $t \in C$ let $B_t = B_{T_C}(t) \setminus \{U \in B_{T_C}(t) : U \setminus C \neq \emptyset\}$. Now we describe a family $B(t)$, for any $t \in \bar{A}$. We make the construction from the right side of t. Consider the following cases:

1. There exists $a > t$ such that $(t,a) \in \ell(B)$ and $t \notin A$. Then we put

$$B^+(t) = B_{T_{(t,a)}}(t).$$

2. $t \in A$. Then we put:

$$B^+(t) = \{[t, t + \tau_n) : n=1,2,\ldots\},$$

where $\{\tau_n\}_{n=1}^{\infty}$ is some decreasing sequence tending to zero.

3. $t \notin A$ and there exists $a > t$ such that $(t,a) \subset A$. Since t is a Darboux point for H with respect to A, there exists a sequence $\{t_n\}_{n=1}^{\infty} \subset (t,a)$, $t_n \searrow t$ and $\lim_{n\to\infty} h(t_n) = h(t)$ for every $h \in H$. For every t_n let $\{\tau_k^n\}_{k=1}^{\infty}$ be a sequence such that $\tau_k^n \underset{k\to\infty}{\searrow} 0$. We additionally assume that $(t_n-\tau_1^n,\ t_n+\tau_1^n) \subset (t,a)$. Then we put:

$$B^+(t) = \{\{t\} \cup \bigcup_{n=m}^{\infty} (t_n-\tau_m^n,\ t_n+\tau_m^n) : m=1,2,\ldots\}.$$

4. $t \notin A$ and t is neither a left endpoint of any component of B nor of A. For every $n=1,2,\ldots$ let $\delta_n > 0$ denote a number such that conditions (*) of Definition 2 are fulfilled for $\eta = \frac{1}{n}$. Of course, we may assume $\delta_n \searrow 0$. Let ℓ_n^+ denote the set of all component C of the set A such that $C \cap (t+\delta_{n+1}, t+\delta_n) \neq \emptyset$, $C \subset (t, t+\delta_1)$ and $C \cap (t, t+\delta_{n+1}) = \emptyset$. Furthermore for every

$C \in \ell_n^+$ let $t_C \in \underset{h\ H}{}\ h^{-1}((h(t) - \frac{1}{n}, h(t) + \frac{1}{n})) \cap C$ and let $\{\tau_n^C\}_{n=1}^{\infty}$ be an arbitrary decreasing sequence converging to zero. Assume:

$$B^+(t) = \{\{t\} \cup \bigcup_{n=m}^{\infty} (\bigcup_{C\in\ell_n^+} (t_C-\tau_n^C, t_C+\tau_n^C)): m=1,2,\dots,\}.$$

Moreover, we may assume that for every $C \in \ell_n^+$, $t+\delta_{n+1} < t_C-\tau_1^C$. Analogously, we define the families "neighbourhoods from left side of t" : $B^-(t)$. Finally we put:

$$B(t) = \{U^+ \cup U^- : U^+ \in B^+(t) \wedge U^- \in B^-(t)\}.$$

Let T be the topology generated by the neighbourhood system $\{B(t)\}_{t\in R}$ ([4]). Of course T is finer than the natural topology of R.

Now we shall show that $C(T)(H)$ is a ring of Darboux functions. Let $k \in C(T)(H)$. Then

$$(2) \qquad k = f_o + f_1 h_{1,1}^{s_{1,1}} \dots h_{1,k_1}^{s_{1,k_1}} + \dots + f_p h_{p,1}^{s_{p,1}} \dots h_{p,k_p}^{s_{p,k_p}},$$

where $f_i \in C(T)$ $(i=0,1,\dots,p)$, $h_{j,1} \in H$ and $s_{j,p}$ are natural numbers.

It is obvious that k is continuous at every point of A (in the natural topology). Let $C \in \ell(B)$ and $t \in \bar{C}$. Let $\{t_{\sigma'}^{'+}\}_{\sigma' \in \Sigma'}$ and $\{t_{\delta'}^{'-}\}_{\delta'\in\Delta'}$ denote the subnets of $\{t_\sigma^+\}_{\sigma\in\Sigma}$ and $\{t_\delta^-\}_{\delta\in\Delta}$ resp. satisfying (1), for every function $h_{j,1}$ from (2). Clearly there exist sequences $\{t_n^+\}_{n=1}^{\infty}$, $\{t_n^-\}_{n=1}^{\infty}$, with elements in $\{t'_{\sigma'}\}_{\sigma'\in\Sigma'}$ and $\{t'^{-}_{\delta'}\}_{\delta'\in\Delta'}$ resp. such that $t_n^- \nearrow t \swarrow t_n^+$ (in the natural topology) and $\lim\limits_{n\to\infty} h_{j,l}(t_n^-) = h_{j,l}(t) = \lim\limits_{n\to\infty} h_{j,l}(t_n^+)$, for every $h_{j,l}$ from (2).

Thus, $\lim\limits_{n\to\infty} (h_{1,1}^{s_{1,1}} \ldots h_{1,k_1}^{s_{1,k_1}})(t_n^{\pm}) = (h_{1,1}^{s_{1,1}} \ldots h_{1,k_1}^{s_{1,k_1}})(t)$.

From the construction of the topology T ($\{t_n^{\pm}\}_{n=1}^{\infty}$ converge to t, in T) and from T-continuity of f_1 we infer that $\lim\limits_{n\to\infty} f_1(t_n^{\pm})=f_1(t)$. Applying similar reasoning to remaining elements from (2) we see that $\lim\limits_{n\to\infty} k(t_n^{\pm}) = k(t)$. According to the obvious fact that $k \in B_1$ and the theorem of Young (see, for example [1],[12]) we infer that $k_{|\bar{C}}$ is a Darboux function.

We shall prove now that k is a Darboux function. Suppose to the contrary that there exist $a < b$ and $\alpha \in (k(a), k(b))$ such that

$$(3) \qquad k^{-1}(\alpha) \cap (a,b) = \emptyset.$$

Assume that $k(a) < k(b)$. If $a \in A \cup B$ or a is a left endpoint of some component $C \in \mathcal{L}(B)$ we put $a_1 = a$. In the remaining cases let a_1 be an element of A such that

$$(4) \qquad a_1 \in [a,b) \cap k^{-1}((-\infty,\alpha)).$$

Notice that

$$(5) \qquad \{x > a_1 : [a_1,x) \subset k^{-1}((-\infty,\alpha))\} \neq \emptyset.$$

In fact if $a_1 \in A$, then (5) is true by the continuity of k at this point. If $a_1 \in B$ or a_1 is a left endpoint of some component $C \in \mathcal{l}(B)$, then according to (3) and the fact that k is a Darboux function in some right hand neighbourhood a_1 we have (5).

Let $\xi = \sup \{x > a_1 : [a_1,x) \subset k^{-1}((-\infty,\alpha))\} < b$. Of course $[a_1,\xi) \subset k^{-1}((-\infty,\alpha))$. Suppose that $k(\xi) > \alpha$. Since $k_{|\bar{C}}$ has the Darboux property, for each $C \in \mathcal{l}(B)$ $\xi \notin C$ nor ξ is a right endpoint of some component $C \in \mathcal{l}(B)$. If $\xi \in \bar{A}$ and ξ is not a right endpoint of any component of B, then we find an element $\xi^* \in (a_1,\xi) \cap k^{-1}((\alpha,+\infty))$, which is impossible.

Suppose now that $k(\xi) < \alpha$. Of course $\xi \notin A$. For each $(p,q) \in \mathcal{l}(B)$, $\xi \notin [p,q)$ because $k_{|[p,q]}$ has the Darboux property. So let $\xi \in \bar{A} \setminus A$ with ξ not a left endpoint of any component of B. Let $U \subset (a,b)$ be an element of $B(\xi)$ such that every component of U, which lies to the right of ξ, has nonempty intersection with $k^{-1}((-\infty,\alpha))$. Let $\delta_\xi = \sup \{x > \xi : x \in U\}$. Since for each $C \in \mathcal{l}(A)$, $k_{|C}$ has the Darboux property, $A \cap (\xi,\xi+\delta_\xi)$ $k^{-1}((-\infty,\alpha))$ and so, according to (3), we have:

$$(6) \qquad \bar{A} \cap (\xi,\xi + \delta_\xi) \subset k^{-1}((-\infty,\alpha)).$$

Let S be an arbitrary component of the set $B \cap (\xi, \xi + \delta_\xi)$. Then $\bar{S}$ includes some points of $\bar{A} \cap (\xi,\xi + \delta_\xi)$. From this, (6), (3) and the property of Darboux of $k_{|\bar{S}}$ we have that $S \subset k^{-1}((-\infty,\alpha))$ and consequently $B \cap (\xi, \xi + \delta_\xi) \subset k^{-1}((-\infty,\alpha))$, and so (according to (6)) $(\xi, \xi + \delta_\xi) \subset k^{-1}((-\infty,\alpha))$. The last inclusion contradicts the definition of ξ. According to (3), $k(\xi) \neq \alpha$. The obtained contradiction (k is not defined at ξ) proves that k is a Darboux function.

According to the fact that every function $k \in C(T)(H)$ is a Darboux function we have that H and $C(H)$ are families of Darboux functions, and consequently T is a connected topology.

Let $\mathcal{T}$ be the class of all possible topologies (for different $A \in \mathfrak{A}$) described above. We shall show that $C(\mathcal{T})$ is the complete class of fundamental rings for H with respect to the conditions $(\mathbb{D})$.

Let $\hat{H}$ be a family of Darboux functions compatible to H with respect to the conditions $(\mathbb{D})$ and let $\mathbb{K}$ be a ring of Darboux functions fulfiling $(\mathbb{D})$ with respect to $A_* \subset C_{\mathbb{K}}$ and containing $H \cup \hat{H}$. Let $C \in \mathcal{C}(R \setminus \bar{A}_*)$. Then $f_{|\bar{C}} \in B_1$ for $f \in \mathbb{K}$. Let $\mathcal{H}$ denote the class of all finite subfamilies of functions belonging to $\mathbb{K}$. Let $H_* \in \mathcal{H}$. It is easy to see, that

for every $x \in \bar{C}$ there exist sequences $\{x^+_{H_*,n}\}, \{x^-_{H_*,n}\} \subset C$ such that $x^-_{H_*,n} \nearrow x \swarrow x^+_{H_*,n}$ and
$$\lim_{n\to\infty} h(x^-_{H_*,n}) = h(x) = \lim_{n\to\infty} h(x^+_{H_*,n}), \quad \text{for every } h \in H_*.$$

(Of course, if x is an endpoint of the segment $\bar{C}$, then there exists only one such sequence converging to x from one side). Let $\Sigma^+ = \Sigma^- = \{(H_*,n) : H_* \in \mathcal{H} \wedge n=1,2,\ldots\}$. In Σ^+ and Σ^- we define the directing relations $\rightarrow\!\!\!\!\!3$ and $\rightarrow\!\!\!\!\!33$ resp., in the following way:

let $\sigma_i = (H^i_*,n_i) \in \Sigma^+$ $(i=1,2)$. Then $\sigma_1 \rightarrow\!\!\!\!\!3\, \sigma_2$ if and only if $x^+_{H^1_*,n_1} \geq x^+_{H^2_*,n_2}$

and

let $\sigma_i = (H^i_*, n_i) \in \Sigma^-$ $(i=1,2)$. Then $\sigma_1 \twoheadrightarrow \sigma_2$ if and only if $x^-_{H^1_*,n_1} \leq x^-_{H^2_*,n_2}$

Let $t^+_\sigma(x) = x^+_{H_*,n}$ $(t^-_\sigma(x) = x^-_{H_*,n})$ be the element of $C \setminus \{x\}$ assigned to $\sigma = (H_*,n) \in \Sigma^+$ $(\sigma = (H_*,n) \in \Sigma^-)$. Then the net $\{t^+_\sigma(x)\}_{\sigma\in\Sigma^+}$ $(\{t^-_\sigma(x)\}_{\sigma\in\Sigma^-})$ is decreasing (increasing) and converges to x. Moreover, for every finite set of functions $H_o \subset \mathbb{K}$ there exist the subnets $\{t'^+_{\sigma'}(x)\}_{\sigma'\in\Sigma^{+\prime}}$ and $\{t'^-_{\sigma'}(x)\}_{\sigma'\in\Sigma^{-\prime}}$ such that

$$\lim_{\sigma'\in\Sigma^{\pm\prime}} h(t'^{\pm}_{\sigma'}(x)) = h(x) \qquad \text{for} \quad h \in H_o.$$

This means that we can from the topology T_C, by means of the above nets, as at the beginning of this proof. Similar reasoning can be applied to every component $C \in \ell(R \setminus \bar{A}_*)$. Notice that taking advantage of the method described in first part of this proof we can define the topology T_* for $H \cup \hat{H}$ (instead of A we use A_*). It is easy to see that $T_* \in \mathcal{T}$, which means that $C(T_*)$ is the fundamental ring for $H \cup \hat{H}$.

The above Theorem makes it possible to form fundamental rings, in particular - rings of Darboux functions, for some classes of functions. We may formulate the following additional problems:

Problem 1. Characterize the classes of Darboux functions for which there exist fundamental rings.

Problem 2. Give necessary and sufficient conditions under which families of Darboux functions H_1 and H_2 have a common fundamental ring.

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