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# APPROXIMATION OF INTEGRABLE, APPROXIMATELY CONTINUOUS FUNCTIONS ON $(0, 1)^n$ BY NONDECREASING FUNCTIONS

For $n \geq 1$, let $\Omega$ denote the open unit n–cube, $(0, 1)^n$. Let $\mu$ denote Lebesgue measure, let $\Sigma$ consist of the Lebesgue measurable subsets of $\Omega$, and let $L_p = L_p(\Omega, \Sigma, \mu)$, $p \geq 1$. Let $A_p$ consist of the approximately continuous functions in $L_p$ and let $M_p$ consist of the strictly right continuous functions in the equivalence classes in $L_p$ which contain nondecreasing functions. Finally, let $A_{1+} = \cup_{p>1} A_p$.

When $f \in A_1$, it is known [1] that there is a unique best $L_1$–approximation, $f_1$, to f in $M_1$. Theorem 7 below asserts that if $f^n \in A_1$, $n \geq 1$, and $f^n \to f \in A_1$ in $L_1$–norm, then $\|f_1^n - f_1\|_1 \to 0$. If $f \in L_p$, $p > 1$, then there is a unique function $f_p \in M_p$ with $\|f - f_p\|_p = d_p(f, M_p) = \inf_{h \in M_p} \|f - h\|_p$. According to Theorem 5, $f_p \to f_1$ a.e. and in $L_1$–norm as $p \to 1$. When $n = 1$, it is shown after Theorem 7 that $f_p$ converges uniformly to $f_1$ as $p \to 1$ on each closed subinterval of $\Omega$; however, to contrast with the case when f is bounded, examples are given to show that $f_p$ need not be continuous when $p > 1$ ($f_1$ is continuous when $n = 1$) and to show that $f_p$ need not be bounded when $f_1 \equiv 0$.

The case of bounded, approximately continuous functions, f, on the closed interval [0, 1] is considered in [2]; continuous functions, f, on the closed n–cube $[0, 1]^n$, $n \geq 1$, are considered in [3]. In each of these cases, there is a unique best $L_1$–approximation $f_1$ by nondecreasing functions, and $f_1$ is continuous.

The proper setting for considering unbounded functions is $\Omega$ (cf. [1]). Existence and uniqueness of $f_1$ for $f \in A_1$ are established in [1]. When $n = 1$, the proof of

Lemma 1 in [2] carries over to imply that $f_1$ is continuous. However, when $n > 1$, $f_1$ need not be continuous, even if f is bounded (cf. [1]).

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \Omega$, $x \leq y$ means $x_i \leq y_i$, $i \leq n$, and $x < y$ means $x_i < y_i$, $i \leq n$. A function $g: \Omega \to R$ is nondecreasing on $\Omega$ if $x \leq y$ implies that $g(x) \leq g(y)$; g is strictly right continuous if $g(x) = \inf_{y>x} f(y)$, $x \in \Omega$. Let M denote the set of nondecreasing functions on $\Omega$.

Let S be a measurable set in $\Omega$. The upper metric density of S at a point p is

$$\lim_{n\to\infty} \sup_{Q} \left[\frac{\mu(S \cap Q \cap \Omega)}{\mu(Q \cap \Omega)}: \mu(Q) < \frac{1}{n}\right],$$

where Q is an open n–cube containing p; the lower metric density is defined similarly. If they are equal, the common value is called the metric density of S at p.

A function from $\Omega$ to R is said to be approximately continuous at a point p if, for every open set G containing f(p), the set $f^{-1}(G)$ has metric density 1 at p. A function f is said to be approximately continuous on $\Omega$ if it is approximately continuous at every point of $\Omega$. Lebesgue points of an integrable function are points of approximate continuity, and for bounded functions, the reverse is true.

When $f \in A_1$, f is the only element of $A_1$ which equals f a.e.. Moreover, there is a one–to–one correspondence between the integrable, strictly right continuous, nondecreasing functions g on $\Omega$ and the classes in $L_1$ which contain a nondecreasing function; each element of $M_1$ represents its class in $L_1$.

Consider a nondecreasing function g on $\Omega$. For $u \in R^{n-1}$ and $t \in R$, put $p(u, t) = (u,0) + t1_n$, $1_n = (1,\ldots,1) \in R^n$. Then put $g_u(t) = g(p(u,t))$ when $p(u,t) \in \Omega$. For each u, $g_u$ is a nondecreasing function defined on a bounded, open, connected subset (perhaps empty) of R. Consequently, $g_u$ has a countable set of discontinuities. Moreover, if $p(u, t) \in \Omega$ and $g_u$ is continuous at t, then g is continuous at p(u, t). Fubini's theorem permits us to assert that g is continuous a.e. on $\Omega$. Let $C_g$ denote the set of continuity points of g.

We will use the following four lemmas. The fourth is well known, so we omit a proof for it.

**LEMMA 1.** *Let $f \in A_{1+}$. Then $d_p(f, M_p)$ is a nondecreasing function of p and $\lim_{p\to 1} d_p(f, M_p) = d_1(f, M_1)$.*

**Proof:** We know that if $\phi$ is measurable on $\Omega$ and $1 \le r < s$, then $\|\phi\|_r \le \|\phi\|_s \le \infty$. Since $f \in A_{1+}$, there exists $p_f > 1$ with $f \in A_{p_f}$. Consider $p \le p_f$, then $d_p(f, M_p)$ is nondecreasing and $\lim_{p\to 1} d_p(f, M_p) \ge d_1(f, M_1)$. To establish the reverse inequality, let $\epsilon > 0$ and $h \in M_1$ with $\|f - h\|_1 < d_1(f, M_1) + \epsilon$. Let $h_m$ denote the $m^{th}$ truncate of h: $h(x) = (m \wedge h(x)) \vee (-m)$, where $\|f-h_m\|_1 < \|f-h\|_1 + \epsilon$. Since $\lim_{p\to 1}\|f - h_m\|_p = \|f - h_m\|_1$, let q satisfy $\|f - h_m\|_q < \|f - h_m\|_1 + \epsilon$. Then $d_p(f, M_1) \le \|f - h_m\|_p < \|f - h\|_1 + 2\epsilon < d_1(f, M_1) + 3\epsilon$, $1 < p < q$.

**LEMMA 2.** *Suppose $\{f_m\}$ is a pointwise bounded sequence in M. Then there exists a strictly right continuous, nondecreasing function h on $\Omega$ and a subsequence $\{f_{m_k}\}$ which converges to h on $C_h$.*

**Proof:** Let $\mathcal{R}$ denote the set of points in $\Omega$ all of whose coordinates are rational. Let $\{f_{m_k}\}$ converge on $\mathcal{R}$ and put $g(x) = \lim_k f_{m_k}(x)$, $x \in \mathcal{R}$. Extend g to $\Omega - \mathcal{R}$ by putting $g(y) = \inf(g(x){:}\ y \le x \in \mathcal{R}\}$, $y \in \Omega - \mathcal{R}$. Then g is nondecreasing on $\Omega$ and it is straightforward to verify that $f_{m_k} \to g$ on $C_g$. Now, put $h(x) = \inf_{y>x} g(y)$, $x \in \Omega$. Then h is strictly right continuous, $h = g$ on $C_g$, and $C_h = C_g$.

**LEMMA 3.** *Suppose $\{f_i\}$ is a sequence of functions in $M$ and $\{f_i\}$ is a Cauchy sequence in $L_1$. Then there exists $h \in M$ such that $\|f_i - h\|_1 \to 0$ and $f_i \to h$ on $C_h$.*

**Proof:** Let $\phi \in L_1$ with $\|f_i - \phi\|_1 \to 0$. Let $\{f_{i_j}\}$ be a subsequence of $\{f_i\}$ with $\|f_{i_j} - \phi\|_1 < 2^{-j}$. Then $|f_{i_j}| \leq g = |\phi| + \sum_j |f_{i_j} - \phi| \in L_1$. Since an $L_1$–bounded sequence in M is pointwise bounded, Lemma 2 applies. Thus, there exists a (unique) strictly right continuous $h \in M$ with $f_{i_{j_k}} \to h$ on $C_h$; $|h| \leq g$ on $C_h$. The Dominated Convergence Theorem implies that $\|f_{i_{j_k}} - h\|_1 \to 0$, so $\phi = h$ a.e.. To show that $f_i \to h$ on $C_h$, let $x \in C_h$ and put $\lambda(t) = h(x+t1_n)$, $1_n = (1,\ldots,1) \in R^n$. Then $\lambda$ is continuous at zero. Let $\epsilon > 0$ and choose $\delta > 0$ so that $|\lambda(t)-\lambda(0)| < \epsilon$ where $|t| \leq \delta$. Then $\|f_i - h\|_1 > \epsilon\delta^n$ if $|f_i(x) - h(x)| > 2\epsilon$; thus, $f_i(x) \to h(x)$.

**LEMMA 4.** (cf. [6, p. 90]) *Let $\{\phi_i\}$ be a sequence of integrable functions. Suppose that $\phi$ is integrable, $\phi_i \to \phi$ a.e. and $\|\phi_i\|_1 \to \|\phi\|_1$. Then $\|\phi_i - \phi\|_1 \to 0$.*

**THEOREM 5.** *Let $f \in A_{1+}$. Then $f_p$ converges to $f_1$ on $C_{f_1}$ as $p$ decreases to 1 and $\|f_p - f_1\|_1 \to 0$.*

**Proof:** To show that $f_p \to f_1$ on $C = C_{f_1}$, it suffices to consider $g_m = f_{p_m}$, where $f \in A_{p_1}$ and $p_m$ decreases to 1, and show that a subsequence $g_{m_k} \to f_1$ on C. Notice that $\|f-f_p\|_p \leq \|f-0\|_p = \|f\|_p$; thus, $\|f_{p_m}\|_{p_m} \leq \|f\|_{p_m} + \|f-f_{p_m}\|_{p_m} \leq 2\|f\|_{p_m} \leq 2\|f\|_{p_1}$. Consequently, $\{f_{p_m}\}_m$ is pointwise bounded on $\Omega$, so we apply Lemma 2 to obtain $h \in M$ with $g_{m_k} \to h$ on $C_h$. By Fatou's lemma, $\|f-h\|_1 \leq \underline{\lim}\|f-g_{m_k}\|_1$.

Moreover, since $\|f-g_{m_k}\|_1 \le \|f-g_{m_k}\|_{p_{m_k}}$, $\|f-h\|_1 \le \underline{\lim}\|f-g_{m_k}\|_{p_{m_k}} = \lim\|f-g_{m_k}\|_{p_{m_k}} = d_1(f,M_1)$ because of Lemma 1. By the uniqueness of best $L_1$-approximation, $f_1 = h$. It remains to verify that $\|f_p - f_1\|_1 \to 0$ by showing that $\|f_{p_m} - f_1\|_1 \to 0$ when $p_m$ decreases to one as follows. Put $\phi_m = f - f_{p_m}$ and $\phi = f - f_1$. Then $\phi_m \to \phi$ on C and $\|\phi_m\|_1 \to \|\phi\|_1$, so Lemma 4 applies to finish a proof of Theorem 5.

Before proceeding, we note that the uniqueness of $f_1$, established in [1], and Theorem 2 in [5] imply that $\|f_p - f_1\|_1 \to 0$. Consequently, application of Lemma 3 gives another proof of Theorem 5.

We will show that $L_1$-approximation is continuous on the set of integrable, approximately continuous functions. The following simple example from [2] illustrates that approximate continuity is "necessary" to ensure continuity of $L_1$-approximation.

Let $I_E$ denote the indicator function of a subset E of R: $I_E(x) = 1$, $x \in E$, $I_E(x) = 0$, $x \notin E$.

**Example 6.** Put $f^n(x)=1$, $x \in [0, \frac{1}{2}(1 - \frac{1}{n})]$, $f^n(x)=0$, $x \in [\frac{1}{2}, 1]$, and extend $f^n$ to be linear on $[\frac{1}{2}(1 - \frac{1}{n}), \frac{1}{2}]$. Put $g^n(x) = f^n(x)$, $x \in [0, \frac{1}{2}] \cup [\frac{1}{2}(1 + \frac{4}{n}), 1]$, $g^n(x) = 1$, $x \in [\frac{1}{2}(1 + \frac{1}{n}), \frac{1}{2}(1 + \frac{3}{n})]$, and extend $g^n$ to be linear on each of $[\frac{1}{2}, \frac{1}{2}(1 + \frac{1}{n})]$, $[\frac{1}{2}(1 + \frac{3}{n}), \frac{1}{2}(1 + \frac{4}{n})]$. Then $f^n \le g^n$, $g^n(0)=f^n(0)$, $\int_0^1 (g^n-f^n) \to 0$, and $g_1^n - f_1^n = g_1^n \equiv 1$. Notice that $f^n \to I_{[0,1/2)}$ pointwise and $g^n \to I_{[0,1/2)}$ pointwise. $I_{[0,1/2)}$ is quasi-continuous and has only one point of discontinuity on [0, 1], so the following theorem is tight.

**THEOREM 7.** *Let $f^m \in A_1$. Suppose $f \in A_1$ and $\|f^m - f\|_1 \to 0$. Then $\|f_1^m - f_1\|_1 \to 0$.*

**Proof:** Suppose, on the contrary, that there exists $\epsilon > 0$ and a subsequence, $\{f^{m_k}\}$, such that $\|f_1^{m_k} - f_1\|_1 > \epsilon$. We will show that this supposition leads to a contradiction. A sequence of relabelings permits us to make some additional suppositions. First, we suppose $\|f_1^m - f_1\|_1 > \epsilon$. Since $\|f^m - f\|_1 \to 0$, there exists a subsequence with $\|f^{m_k} - f\|_1 < 2^{-k}$. Thus, $|f^{m_k}| \le |g| = |f| + \sum_k |f^{m_k} - f| \in L_1$; moreover, [5, Lemma 3], $-g_1 \le f_1^{m_k} \le g_1$. Hence, we suppose $|f^m| \le g$, so $|f_1^m| \le g_1$. Next, we suppose $f^m \to f$ a.e.. Now we repeat the argument in Theorem 5 to find a subsequence $f_1^{m_k}$ which converges to $h \in M_1$ on $C_h$. Finally, suppose $f_1^m \to h$ on $C_h$. By the Dominated Convergence Theorem, $\|f_1^m - h\|_1 \to 0$, so $\|f^m - f_1^m\|_1 \to \|f - h\|_1$. Also, $\|f^m - f_1\|_1 \to \|f - f_1\|_1$. Thus, since $\|f^m - f_1^m\|_1 \le \|f^m - f_1\|_1$, $\|f - h\|_1 \le \|f - f_1\|_1$. By the uniqueness of best approximation, $h = f_1$; hence, we have a contradiction.

Henceforth, suppose $n = 1$ and $f \in A_{1+}$. To show that $f_p \to f_1$ uniformly on $[a, b] \subset \Omega$, suppose on the contrary that there is $\epsilon > 0$ and a sequence $p_m$ decreasing to one with $\|f_{p_m} - f_1\|_\infty \ge \epsilon$ on $[a, b]$. As in the proof of Theorem 7, Helly's Theorem gives us a subsequence $g_k = f_{p_{m_k}}$ that converges pointwise to a nondecreasing function $g$ on $[a, b]$; as before, $g = f_1$ a.e. on $[a, b]$. But $f_1$ is continuous (cf. [2, Lemma 1]), so $g = f_1$ on $[a, b]$. Moreover, because $g$ is continuous, we can conclude that $g_k \to g$ uniformly on $[a, b]$. This contradiction establishes the uniform convergence of $f_p$ to $f_1$ on $[a, b]$.

Two examples follow. Example 8 shows that $f_p$ need not be continuous. Example 9 shows that $f_p$ need not converge uniformly to $f_1$; indeed, $f_1 \equiv 0$ and $\lim_{t\to 1} f_p(t) = \infty$, $p > 1$ in this example.

**Example 8.** Consider the step function $\phi$ defined on $[0, a + b]$ by $\phi = hI_{[0,a]}$. For $p > 1$, the best $L_p$–approximation to $\phi$ by nondecreasing functions is given by the constant function $\phi_p = h_p I_{[0,a+b]}$, where

$$h_p = \frac{h}{\left[1 + \left[\frac{b}{a}\right]^{\left[\frac{1}{p-1}\right]}\right]};$$

$h_1 \equiv 0$ if $a < b$.

For $n \geq 2$, put $a_n = K(n^{(p+1)}\ln^2 n)^{-1}$, $b_n = K(n^2 \ln^2 n)^{-1}$ and $c_n = a_n + b_n$, where $\sum_{n\geq 2} c_n = \frac{1}{2}$. Put $h_n = n + 1$, $u_n = \frac{1}{2} + \sum_{k\geq n} c_k$ and define $\phi$ on $\Omega$ by $\phi = \sum_{n\geq 2} h_n I_{[u_{n+1}, u_{n+1}+a_n]}$; $\phi \in L_p$.

On $[u_{n+1}, u_n]$ the best nondecreasing $L_p$–approximation to $\phi$ is the constant function $I_{[u_{n+1},u_n]}$; hence, $\phi_p = I_{[\frac{1}{2},1]}$.

Since the map $g \to g_p$ is order preserving on $L_p$ [4], we can modify $\phi$ to obtain $f \in A_p$ with $f(x) = 0$, $x \in (0, \frac{1}{2}]$ and $f \geq \phi$. Then $f_p(x) = 0$, $x \in (0, \frac{1}{2})$ and $f_p(x) \geq 1$, $x \in [\frac{1}{2}, 1)$.

**Example 9.** Let $a_n$ remain as in Example 8. Put $t_n = (\frac{n}{\ln n - 1})$, $n > 1$, and modify $b_n$:

$$b_n = K \exp\{(\ln t_n)^{-1/2} \ln t_n\}/[n^{(p+1)}\ln^2 n].$$

Again put $c_n = a_n + b_n$ and specify $K$ by the equation $\sum_{n\geq 2} c_n = \frac{1}{2}$. Put $u_n = 1 - \sum_{k\geq n} c_k$, $h_n = n$ and $\phi = \sum_{n\geq 2} h_n I_{[u_n, u_n + a_n]}$. Then $\phi \in L_p$. On

$[u_n, u_{n+1})$, the best nondecreasing $L_t$-approximation to $\phi$ is $\epsilon_n(t)I_{[u_n,u_{n+1}]}$, where $\epsilon_n(t)$ is a nondecreasing function of t on (1, p]. Let's look at $\epsilon_n(p_n)$ for $p_n - 1 = (\ln t_n)^{-1/2}$: $\epsilon_n(p_n) = \frac{n}{(1 + t_n)} = \ln n \to \infty$. Hence $\lim_{x\to 1}\phi_t(x) = \infty$, $1 < t \leq p$. Again, modify $\phi$ to obtain $f \in A_p$ with $f \geq \phi$ such that $f_1 \equiv 0$ and $\lim_{x\to 1}f_t(x) = \infty$, $1 < t \leq p$.

## REFERENCES


1. R. B. Darst and S. S. Fu, "Best $L_1$-approximation of $L_1$-approximately continuous functions on $(0, 1)^n$ by nondecreasing functions", ***Proc. Amer. Math. Soc.***, 97 (1986), 262–264.
2. R. B. Darst and R. Huotari, "Best $L_1$-approximation of bounded approximately continuous functions on [0, 1] by nondecreasing functions", ***J. Approx. Theory*** 43 (1985), 178–189.
3. R. B. Darst and R. Huotari, "Monotone $L_1$-approximation on the unit n-cube", ***Proc. Am. Math. Soc.*** 95 (1985), 425–428.
4. D. Landers and L. Rogge, "Isotonic approximation in $L_s$", ***J. Approx. Theory*** 31 (1981), 199–223.
5. D. Landers and L. Rogge, "Natural choice of $L_1$-approximants", ***J. Approx. Theory*** 33 (1981), 268–280.
6. H. L. Royden, ***Real Analysis***, 2nd Edition, Macmillan, New York, 1968.


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