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INTEGRALS OF LUSIN AND PERRON TYPE

In the first part of the present paper we study the relations between Lee's LDG- and LPG-integrals ([7]) and conditions [M], [$\overline{M}$] and [$\underline{M}$]. Also we give similar results for other integrals introduced here. These results are then used to obtain change of variable formulas.

In the second part we introduce an integral of Perron type which is equivalent to the Foran integral.

In what follows we refer to the following classes of functions: $\mathcal{C}$, (N), N^{∞}, $N^{-\infty}$, $N^{+\infty}$, [$\underline{M}$], [$\overline{M}$], [M], [$\underline{M}_*$], [$\overline{M}_*$], [M_*], A(N), B(N), $\mathcal{F}$, $\mathcal{B}$, $\mathcal{E}$, AC, $\overline{AC}$, $\underline{AC}$, VB, ACG, $\overline{ACG}$, $\underline{ACG}$, VBG, D, DB_1, B_1^*, DB_1^*, uCM, lCM, CM. For all of these see [4]. Let $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ denote the semilinear space generated by the classes of functions $\mathfrak{a}_1$ and $\mathfrak{a}_2$.

Some properties of the LPG- and LDG-integrals. Change of variables.

<u>Definition 1</u>. [8]. A function $f : [0,1] \to \mathbb{R}$ is said to be $\overline{AC}_*$ on a set $E \subset [0,1]$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum(f(b_k) - f(c_k)) < \varepsilon$ and $\sum(f(c_k) - f(a_k)) < \varepsilon$ for each sequence of nonoverlapping intervals $\{(a_k,b_k)\}$ with endpoints in E, $a_k \le c_k \le b_k$ and $\sum (b_k - a_k) < \delta$. Let $\underline{AC}_* = \{f : -f \in \overline{AC}_*\}$ and $AC_* = \overline{AC}_* \cap \underline{AC}_*$. The classes $\overline{ACG}_*$, $\underline{ACG}_*$ and ACG_* are defined analogously to ACG.

<u>Defnition 2</u>. [9]. A function $f : [0,1] \to \mathbb{R}$ is said to be $(N_g^{\pm\infty})$ if [0,1] is the union of a countable sequence of perfect sets P_j (except perhaps a countable set of points) such that:

1° The set $f(\{x \in P_j : f|'_{P_j}(x) = +\infty\})$ has measure 0 for each j;

2° On each perfect subset T_j of P_j, $f(x)$ satisfies an analogous condition. Let $(N_g^{-\infty}) = \{f : -f \in (N_g^{+\infty})\}$ and let $(N_g) = (N_g^{+\infty}) \cap (N_g^{-\infty})$.

__Remark 1.__ Definition 2 can be simplified as follows:

__Definition 2'.__ A function $f: [0,1] \longrightarrow \mathbb{R}$ is said to be $(N_g^{+\infty})$ if $|f(\{x \in P : f|'_P(x) = +\infty\})| = 0$ for each perfect subset P of $[0,1]$.

__Proof.__ Clearly Definition 2' implies Definition 2. Conversely, let P be a perfect subset of $[0,1]$ and let P_j be the perfect sets of Definition 2. Let $A = \{x \in P : f|'_P(x) = +\infty\}$, $A_j = A \cap P_j \subset P \cap P_j = T_j$. Let $B_j = \{x \in T_j : f|'_{T_j}(x) = +\infty\}$. Clearly $A_j \subset B_j$. Since by 2°, $|f(B_j)| = 0$, it follows that $|f(A)| = 0$.

__Proposition 1.__ __Let__ $F : [0,1] \longrightarrow \mathbb{R}$, $F \in (N_g^{+\infty})$. __Then__ $F \in [\overline{M}]$ __on__ $[0,1]$. __If__ $F \in B_1^*$, __then conditions__ $(N_g^{+\infty})$ __and__ $[\overline{M}]$ __are equivalent.__

__Proof.__ Let $F \in (N_g^{+\infty})$ and let P be a perfect subset of $[0,1]$ such that $F|_P \in VB \cap \mathcal{C}$. We define $F_1(x) = F(x)$, $x \in P$, and linearly on the closure of each interval contiguous to P. Clearly $F_1 \in \mathcal{C} \cap VB \cap N^{+\infty}$ on $[0,1]$. By Corollary 2 of [4], $F_1 \in \overline{\overline{AC}}$ on $[0,1]$. Hence $F \in \overline{\overline{AC}}$ on P.

For the second part it suffices to show that $[\overline{M}] \cap B_1^* \subset (N_g^{+\infty})$. Let P be a perfect subset of $[0,1]$. Since $F \in B_1^*$, $P = (\cup P_j) \cup \{a_k\}$ such that P_j are perfect subsets of P and $F|_{P_j} \in \mathcal{C}$. We define $F_j(x) = F(x)$, $x \in P_j$ and linearly on the closure of each interval contiguous to $P_j \subset [0,1]$. Then $F_j \in \mathcal{C} \cap [\overline{M}]$ on $[0,1]$ and by Theorem 6 of [4], $F_j \in N^{+\infty}$ on $[0,1]$. Let $E_j^{+\infty} = \{x \in P_j : F|'_P(x) = +\infty\}$. If $x \in E_j^{+\infty}$ is a bilateral accumulation point for P_j, then $F_j'(x) = +\infty$. Hence $|F(E_j^{+\infty})| = 0$. Let $E^{+\infty} = \{x \in P : F|'_P(x) = +\infty\}$. Since $E^{+\infty} \cap P_j \subset E_j^{+\infty}$, $|F(E^{+\infty})| = 0$. Hence $F \in (N_g^{+\infty})$.

__Proposition 2.__ __Let__ Q __be a perfect subset of__ $[0,1]$, $a = \inf(Q)$, $b = \sup(Q)$ __and let__ $F : [a,b] \longrightarrow \mathbb{R}$ __be a bounded function. Then the following statements are equivalent:__ 1° $F \in$ __AC__ $\cap\ VB_*$ __on__ Q; 2° __There exist__ $F_1, F_2 : [a,b] \longrightarrow \mathbb{R}$ __such that__ $F = F_1 + F_2$, $F_1 \in AC_*$ __on__ Q, F_2 __is increasing and__ $F_2'(x) = 0$ a.e. __on__ $[a,b]$; 3° F __is__ AC_* __on__ Q.

Proof. We first show that 1° implies 2°. We define $f(x) = F(x)$, $x \in Q$, and linearly on the closure of each interval contiguous to Q. Then $F \in \underline{AC}$ on $[a,b]$. By note 13, page 169, of [9], $f = f_1 + f_2$, where $f_1 \in AC$, f_2 is increasing on $[a,b]$ and $f_2'(x) = 0$ a.e. on $[a,b]$. Let $F_2(x) = f_2(x)$ on $[a,b]$ and $F_1(x) = F(x) - f_2(x)$ on $[a,b]$. Then $F_1 \in VB_*$ on Q and $F_1(x) = f_1(x)$ on Q. Hence F_1 is also AC on Q. By Theorem 8.8, page 233 of [11], F_1 is AC_* on Q.

We now show that 2° implies 3°. Let $F = F_1 + F_2$ such that 2° is satisfied. Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that for each sequence of nonoverlapping intervals $\{I_k\} = \{(a_k,b_k)\}$ with $\sum(b_k - a_k) < \delta$, we have $\sum_k O(F_1; [a_k,b_k]) < \varepsilon$. For $a_k \le c_k \le b_k$ we have $\sum_k (F_1(c_k) - F_1(a_k)) \ge -\varepsilon$. Since $\sum_k (F_2(c_k) - F_2(a_k)) \ge 0$, $\sum_k (F(c_k) - F_1(a_k)) \ge -\varepsilon$. Similarly $\sum_k (F(b_k) - F(c_k)) \ge -\varepsilon$. Hence $F \in \underline{AC}_*$ on Q.

We finally show that 3° implies 1°. Since $F \in \underline{AC}_*$ on Q, $F \in \underline{AC}$ on Q. Hence $F \in VB$ on Q. Since $F \in \underline{AC}_*$ on Q, for $\varepsilon_0 > 0$ there is a natural number k_0 such that for $I_k = [a_k,b_k]$, the intervals contiguous to Q, we have: $\sum_{k=k_0}^{\infty} (F(a_k) - F(c_k)) < \varepsilon_0$ and $\sum_{k=k_0}^{\infty} (F(c_k) - F(b_k)) < \varepsilon_0$, when $c_k \in [a_k,b_k]$. Hence $\sum_{k=k_0}^{\infty} |m_k - F(a_k)| = \sum_{k=k_0}^{\infty} (F(a_k) - m_k) \le \varepsilon_0$ and $\sum_{k=k_0}^{\infty} |F(b_k) - M_k)| = \sum_{k=k_0}^{\infty} (M_k - F(b_k)) \le \varepsilon_0$, where $m_k = \inf\{F(x) : x \in I_k\}$, $M_k = \sup\{F(x) : x \in I_k\}$. We have $\sum_{k=k_0}^{\infty} O(F;I_k) \le \sum_{k=k_0}^{\infty} |m_k - F(a_k)| + \sum_{k=k_0}^{\infty} |F(a_k) - F(b_k)| + \sum_{k=k_0}^{\infty} |F(b_k) - M_k| \le 2\cdot\varepsilon_0 + V(F;Q)$, where $V(F;Q)$ is the variation of F on Q. Since F is bounded on $[a,b]$, $\sum_{k=1}^{\infty} O(F;I_n)$ is convergent. By Theorem 8.5, page 232 of [11], $F_{|Q}$ is VB_*.

Corollary 1. a) _A function_ F _belongs to_ $[\underline{M}_*]$ _on a bounded closed set_ E _if and only if_ F _is_ AC_* _on each closed subset of_ E _on which it is continuous and_ VB_*: b) $B_1^* \cap VBG_* \cap [\underline{M}_*] \subset \underline{ACG}_*$ _on a closed set_ E: c) $D \cap VBG_* \cap [\underline{M}_*] = \underline{ACG}_* \cap DB_1^*$ (_since_ $D \cap VBG_* \subset B_1^*$, _according to Lemma A of_ [4]).

Theorem 1. *Let* $h : [0,1] \to \mathbb{R}$ *be such that* $h \in ([\underline{M}] \cap C) + \underline{AC}$ *and* $h'(x) \geq 0$ a.e. *where* h *is derivable. Then* h *is increasing on* $[0,1]$.

Proof. Let $f \in [\underline{M}] \cap C$ and $g \in \underline{AC}$ such that $h = f + g$. By note 13, page 169 of [9], $g = g_1 + g_2$, with $g_1 \in AC$, g_2 increasing on $[0,1]$ and $g_2'(x) = 0$ a.e. on $[0,1]$. Then $h = f + g_1 + g_2 = h_1 + g_2$. Clearly $h_1 \in [\underline{M}] \cap C$ on $[0,1]$ and $h_1'(x) \geq 0$ a.e. where h_1 is derivable. By Theorem 10 of [4], h_1 is increasing on $[0,1]$. Hence h is increasing on $[0,1]$.

Theorem 2. *Let* $h : [0,1] \to \mathbb{R}$ *be such that* $h \in ((B_1^* \cap [\underline{M}]) \oplus [\underline{ACG}]) \cap uCM$ *and* $h'(x) \geq 0$ a.e. *where* h *is derivable. Then* h *is increasing on* $[0,1]$.

Proof. Let $f \in [\underline{M}] \cap B_1^*$ and $g \in [\underline{ACG}]$ such that $h = f + g$ on $[0,1]$. Then there exists a sequence of intervals $\{I_n\}$ whose union is dense in $[0,1]$ such that $f \in [\underline{M}] \cap C$ on I_n and $g \in \underline{AC}$ on I_n. Let $[a_n,b_n] \subset I_n$. By Theorem 1, h is increasing on $[a_n,b_n]$. Hence h is increasing on I_n. The intervals I_n can be chosen to be maximal open intervals of monotonicity of h. Suppose to the contrary that $\cup I_n \neq (0,1)$ and let $Q = [0,1] - \cup I_n$. Snce $h \in uCM$, Q is a perfect subset of $[0,1]$ (if necessary without 0 and 1). Let $a,b \in Q$ such that $Q \cap (a,b) \neq \emptyset$ and $f_{|Q \cap [a,b]} \in [\underline{M}] \cap C$ and $g_{|Q \cap [a,b]} \in \underline{AC}$. Let $f_1(x) = f(x)$; $g_1(x) = g(x)$; $h_1(x) = h(x)$, $x \in Q \cap [a,b]$. Extend f_1, g_1, h_1 linearly on the closure of each interval contiguous to Q. We have $f_1 \in C \cap [\underline{M}]$ by the proof of [4], Theorem 11, $g_1 \in \underline{AC}$, $h_1 = f_1 + g_1$ on $[a,b]$. If f_1 in Theorem 11 of [4] is replaced by h_1, since condition (i) of Lemma 7 of [4] can be omitted, $h_1'(x) \geq 0$ a.e. where $h_1'(x)$ exists on $[a,b]$. Now by Theorem 1, h_1 is increasing on $[a,b]$. Hence h is increasing on $[a,b]$, a contradiction.

Remark 2. If uCM is replaced by the Darboux property D, then Theorem 2 remains true (since $D \subset uCM$) and in addition h is also continuous.

Let uL denote an upper semilinear space contained in uCM. Let $\ell L = \{F : -F \in uL\}$ and $L = uL \cap \ell L$.

Definition 3. A function M is said to be a LPB (respectively L_0PG; L_*PG) - major function for a function $f : [0,1] \to R$ if: (i) $M(0) = 0$; (ii) $M \in uL$; (iii) $\ell M'_{ap}(x) \geq f(x)$ (resp. $\ell M'(x) \geq f(x)$; $\ell M'(x) \geq f(x)$) a.e. on $[0,1]$; (iv) $M \in [\underline{ACG}]$ (resp. $M \in [\underline{ACG}]$; $M \in [\underline{ACG_*}]$). m is a LPG (resp. L_0PG; L_*PG) - minor function for f if $-m$ is a LPG (resp. L_0PG; L_*PG) major function of F. A function f is LPG (resp. L_0PG; L_*PG) integrable on $[0,1]$ if:

1° f has LPG (resp. L_0PG; L_*PG) major and minor functions on $[0,1]$;

2° for each $\varepsilon > 0$ there exists a LPG (resp. L_0PG; L_*PG) major function M and a LPG (resp. L_0PG; L_*PG) minor function m such that $M(x) - m(x) \leq \varepsilon$, $x \in [0,1]$. Then

$$\text{LPG(resp. } L_0PG;\ L_*PG) \int_0^1 f(x)dx = \inf_M \{M(1)\} = \sup_m \{m(1)\} .$$

Remark 3. By [8] Theorem XVIII, page 252 and Theorem XI, page 245, a function which satisfies $[\underline{ACG_*}]$ (resp. $[\underline{ACG}]$) on $[0,1]$ is derivable (resp. approximately derivable) a.e. on $[0,1]$. Hence in the definition of LPG (resp. L_*PG) condition (iii) can be replaced by (iii'): $M'_{ap}(x) \geq f(x)$ (resp. $M'(x) \geq f(x)$) a.e. on $[0,1]$.

Definition 4. A function $f : [0,1] \to \mathbb{R}$ is said to be LDG (resp. L_0DG; L_*DG) - integrable on $[0,1]$ if there is a function $F \in L \cap [ACG]$ (resp. $F \in L \cap [ACG] \cap \Delta_{a.e.}$; $F \in L \cap \underline{ACG_*}$) such that $F'_{ap}(x) = f(x)$ (resp. $F'(x) = f(x)$; $F'(x) = f(x)$) a.e. on $[0,1]$. In all these cases the integral of f over $[0,1]$ is defined to be $F(1) - F(0)$. ($\Delta_{a.e.} = \{F : [0,1] \to \mathbb{R} : F$ is derivable a.e. on $[0,1]\}$.)

Remark 4. a) The LDG and LPG integrals were introduced by Lee in [7] and he proved that these two integrals are equivalent if uL is closed under uniform convergence. Using Theorem 1 of [7], we can prove that the L_0PG (resp. L_*PG) - integral is equivalent with the L_0DG (resp. L_*DG) - integral.

b) If in Definition 4 L is the class of all approximately continuous functions on $[0,1]$, then the LDG (resp. L_0DG; L_*DG) - integral is in fact the β (resp. β_0; α) - integral of Ridder. (See [10].)

For a function f on $[0,1]$ we define $f^*(x) = f'(x)$ (resp. $f^*_{ap}(x) = f'_{ap}(x)$) where $f'(x)$ (resp. $f'_{ap}(x)$) exists and is finite and 0 elsewhere.

Theorem 3. *Let $\mathcal{A}$ be a class of functions such that $(B_1^* \cap \mathcal{A}) \oplus uL \subset uCM$ on $[0,1]$. Let $F : [0,1] \to \mathbb{R}$ satisfy the following properties:* 1° $F \in (-\mathcal{A}) \cap B_1^*$ *on* $[0,1]$; 2° $F \in [\overline{M}]$ *on* $[0,1]$;

3° F^*_{ap} *(resp.* F^*; F_*) *has a* LPG *(resp.* L_0PG; L_*PG) *- major function* G *on* $[0,1]$. *Then we have that:*

a) F *is* $[\overline{ACG}]$ *(resp.* $[\overline{ACG}]$; $[\overline{ACG_*}]$) *and* $G - F$ *is increasing on* $[0,1]$;

b) *If in addition* $F \in [M] \subset [\overline{M}]$, *then* $F \in [ACG]$ *(resp.* $[ACG]$, $[ACG_*]$) *and* $G - F$ *is increasing on* $[0,1]$.

Proof. Let $H = G - F$. In the first and third case clearly $H'(x) \geq 0$ a.e. where H is derivable. In the second case G is derivable a.e. on $[0,1] - E$, where $E = \{x : F'(x)$ exists and is finite$\}$ by [11], Theorem 7.2, page 230 and Theorem 10.1, page 234. It follows that $H'(x) \geq 0$ a.e. where H is derivable. Clearly $-F \in \mathcal{A} \cap B_1^* \cap [\underline{M}]$. Since $\mathcal{A} \oplus uL \subset uCM$, by Theorem 2, H is increasing on $[0,1]$. Hence $F \in B_1^* \cap [VBG]$ (resp. $B_1^* \cap [VBG]$; $B_1^* \cap VBG_*$).

a) Since $F \in [\overline{M}]$, $F \in [\overline{ACG}]$ (resp. $[\overline{ACG}]$; $[\overline{ACG}] \cap VBG_* = [\overline{ACG_*}]$ by Corollary 1).

b) Since $F \in [M]$, $F \in [ACG]$ (resp. $[ACG]$; $[ACG_*]$).

Remark 5. Theorem 3, a), b), the first and the third case remain true if 3° is replaced by "there exists a function $f : [0,1] \to \mathbb{R}$ such that $f(x) = F'_{ap}(x)$ a.e. where F is approximately derivable (resp. $f(x) = F'(x)$ a.e. where F is derivable) and f has a LPG (resp. L_*PG) - major function on $[0,1]$".

Remark 6. a) In Theorem 3, a), b), $\mathcal{A}$ can be taken to be uL, where: 1) $uL = \mathcal{C}$; 2) $uL = \{F : F$ is approximately continuous$\}$; 3) $uL = \{F : F$ is an exact nth Peano derivative$\}$; 4) $uL = \{F : F$ is an exact nth approximate Peano derivative$\}$. b) In Theorem 3, a), b), $\mathcal{A}$ can be taken to be D and $uL = \mathcal{C}$ (since $D \subset uCM$ and $DB_1^* \oplus \mathcal{C} = DB_1^*$, by [2], Theorem VI, page 474). By Remark 2, $G - F \in \mathcal{C}$. Hence $F \in \mathcal{C}$.

Corollary 2. _A function_ $F \in L$ _(resp._ $F \in D$_) on_ $[0,1]$ _is an indefinite_ LDG _(resp._ $\mathcal{C}$DG) _- integral if and only if_ $F \in B_1^* \cap [M]$ _and there is a function_ $f : [0,1] \to \mathbb{R}$ _which has a_ LPG _(resp._ $\mathcal{C}$PG) _- major function and_ $f(x) = F'_{ap}(x)$ a.e. _where_ F _is approximately derivable._

Proof. The necessity is evident and the sufficiency follows by Remark 5 and Remark 6.

Remark 7. The first part of Remark 5 extends Lemma B of [9] page 176 and the first part of Corollary 2 is an extension of Theorem VII of [9] page 178, since we give up the condition T_2 and in Ridder's results L is the class of all approximately continuous functions. (Also see Proposition 1.)

Theorem 4. _Let_ $F : [0,1] \to \mathbb{R}$, $F \in DB_1 \cap T_2 \cap [M_*]$. _If_ F^* _has a_ $\mathcal{C}_*$PG _- major function, then_ $F \in ACG_* \cap \mathcal{C}$ _on_ $[0,1]$.

Proof. Since $N^\infty = [M_*]$ for Darboux functions (See [4], Theorem 6.) and $\mathcal{C} \cap VBG_* \cap [M_*] = \mathcal{C} \cap ACG_*$, the theorem follows by the second part of Corollary 6 of [4].

Remark 8. Theorem 4 remains true if $T_2 \cap [M_*]$ is replaced by Lusin's condition (N) according to Remark 1,k of [4].

A function F defined on an interval I is said to be strictly increasing$_*$ (resp. decreasing$_*$) on a set $E \subset I$ if for any $x_1, x_2 \in [\inf(E), \sup(E)]$, $x_1 < x_2$, we have $F(x_1) < F(x_2)$ (resp. $F(x_1) > F(x_2)$), provided that at least one of the points x_1, x_2 belongs to E [12]. If the function F is either strictly increasing$_*$ or strictly decreasing$_*$ on a set E, then F is said to be strictly monotone$_*$.

Proposition 3. _A function_ $f : [0,1] \to \mathbb{R}$ _satisfies condition_ $[\overline{M}_*]$ _(resp._ $[\overline{M}]$) _on a closed subset_ E _of_ $[0,1]$ _if and only if_ $f \in AC$ _on any closed subset of_ E _on which it is continuous and strictly increasing$_*$ (resp. continuous and strictly increasing)._

Proof. Let $P = \overline{P} \subset E$ be such that $f \in VB_* \cap \mathcal{C}$ on P. Let $a = \inf(P)$, $b = \sup(P)$ and $F(x) = f(x)$, $x \in P$. Extending F linearly on each interval contiguous to P we have F defined, continuous and VB on $[a,b]$. Let $E_1^{+\infty} = \{x \in P : F'(x) = +\infty\}$; $E^{+\infty} = \{x \in P : f'(x) = +\infty\}$; $Z = \{x \in P : f'(x)$

does not exist finite or infinite}. Then by Theorem 7.2 of [11], page 230 it follows that $|f(Z)| = 0$. Clearly $E_1^{+\infty} \subset E^{+\infty} \cup Z$. Let $E_n = \{x \in P : (f(x+h) - f(x))/h > 1,\ 0 < |h| \leq 1/n\}$ and let $E_{in} = [i/n, (i+1)/n] \cap E_n$. Since f is continuous on P, $(f(y) - f(x))/(y-x) \geq 1$, for $y \in [\inf(E_{in}), \sup(E_{in})]$, $x \in \overline{E}_{in}$, $x \neq y$. Hence f is strictly increasing$_*$ on $\overline{E}_{in}$. By hypothesis, $f \in AC$ on $\overline{E}_{in}$. Since $|E^{+\infty}| = 0$ (See [11], Theorem 4.4, page 270.), $|f(E^{+\infty})| = 0$ and $|F(E_1^{+\infty})| = 0$. Hence $F \in \mathcal{C} \cap VB \cap N^{+\infty} = \overline{AC}$ on $[a,b]$ by Corollary 2 of [4] and $f \in [\overline{M}_*]$ on E. The converse is evident.

We prove the second part. Suppose that $f \in VB \cap \mathcal{C}$ on P. Let a, b, F and $E_1^{+\infty}$ be defined as above. Let $E_n^1 = \{x \in P : (F(x+h) - F(x))/h > 1,\ 0 < |h| \leq 1/n\}$ and let $E_{in}^1 = [i/n, (i+1)/n] \cap E_n^1$. Since $f \in \mathcal{C}$ on P, $(f(y)-f(x))/(y-x) \geq 1$, for $x,y \in \overline{E}_{in}^1$, $x \neq y$. Hence f is strictly increasing on $\overline{E}_{in}^1$. By hypothesis $f \in AC$ on $\overline{E}_{in}^1$. It follows that $|F(E_1^{+\infty})| = 0$ and $F \in VB \cap N^{+\infty} \cap \mathcal{C} = \overline{AC}$ on $[a,b]$. Hence $f \in [\overline{M}]$ on E. The converse is evident.

<u>Corollary 3</u>. <u>A</u> <u>function</u> $f : [0,1] \to \mathbb{R}$ <u>satisfies</u> <u>condition</u> $[M_*]$ (<u>resp.</u> $[M]$) <u>on</u> <u>a</u> <u>closed</u> <u>subset</u> E <u>of</u> $[0,1]$ <u>if</u> <u>and</u> <u>only</u> <u>if</u> $f \in AC$ <u>on</u> <u>any</u> <u>closed</u> <u>subset</u> <u>of</u> E <u>on</u> <u>which</u> <u>it</u> <u>is</u> <u>continuous</u> <u>and</u> <u>strictly</u> <u>monotone</u>$_*$ (<u>resp.</u> <u>continuous</u> <u>and</u> <u>strictly</u> <u>monotone</u>).

<u>Proposition 4</u>. <u>Let</u> $g : [0,1] \to \mathbb{R}$, $f : g([0,1]) \to \mathbb{R}$.

a) <u>If</u> $f,g \in [M_*] \cap D$, <u>then</u> $f \circ g \in [M_*] \cap D$;

b) <u>If</u> $f,g \in [M] \cap B_1^*$, <u>then</u> $f \circ g \in [M] \cap B_1^*$.

<u>Proof</u>. a) Let $F = f \circ g$. Then $F \in D$. Let $P = \overline{P} \subset [0,1]$ such that F is strictly increasing$_*$ on P, for example. By Lemma A of [4], F is continuous on P. Hence $Q = F(P)$ is a compact set. Clearly $F|_P$, $g|_P$ and $f|_Q$ are injective. We prove that g is strictly monotone$_*$ on P. Suppose on the contrary that there exist $x_1 < x_2 < x_3$, $x_1, x_3 \in P$, such that $g(x_2)$ does not belong to the interval with endpoints $g(x_1)$ and $g(x_3)$. Suppose, for example, that $g(x_1) < g(x_3)$. Then $g(x_2)$ does not belong to $(g(x_1),g(x_3))$. Hence we have two possibilities: (i) $g(x_2) \leq g(x_1) < g(x_3)$ or (ii) $g(x_1) < g(x_3) \leq g(x_2)$. We treat only the case (i). Since $g \in D$, there exists $c \in [x_2,x_3)$ such that $g(c) = g(x_1)$. Then $F(c) - F(x)$, a

contradiction. By Corollary 3 $g \in AC$ on P. Clearly f is strictly monotone on Q. Suppose that f is strictly increasing on Q. We prove that f is strictly increasing$_*$ on Q. Let $y_1 < y_2 < y_3$, $y_1,y_3 \in Q$. Let $x_1,x_3 \in P$ such that $g(x_1) = y_1$, $g(x_3) = y_3$. Then $x_1 < x_3$. Since $g \in D$, there exists $x_2 \in (x_1,x_3)$ such that $g(x_2) = y_2$. Since F is strictly increasing$_*$ on P, $F(x_1) < F(x_2) < F(x_3)$. Hence $f(y_1) < f(y_2) < f(y_3)$. By Corollary 3, $f \in AC$ on Q. It follows that $F \in (N)$ on P. By Theorem 6.7 of [11] page 227, F is AC on Q.

b) Clearly $F \in B_1^*$ on $[0,1]$. Let $K = \bar{K} \subset [0,1]$ be such that F is continuous and strictly monotone on K. Since $f,g \in B_1^*$, $K = \cup K_n$ such that $g|_{K_n} \in \mathcal{C}$, $f|_{g(K_n)} \in \mathcal{C}$. Let $g_n(x) = g(x)$, $x \in K_n$; g_n is linear on the closure of each interval contiguous to K_n; $F_n(x) = F(x)$, $x \in K_n$; F_n is linear on the closure of each interval contiguous to K_n; $f_n(x) = f(x)$, $x \in g(K_n)$; f_n is linear on the closure of each interval contiguous to $g(K_n)$. Then g_n, F_n and f_n are continuous. Hence g_n, F_n and f_n are strictly monotone. (See the proof of a).) Therefore $g|_{K_n}$ and $f|_{g(K_n)}$ are continuous and strictly monotone. By Corollary 3 $g|_{K_n} \in AC$ and $f|_{g(K_n)} \in AC$. Hence $F|_{K_n} \in (N)$. By Theorem 6.7 of [11] page 227, $F \in AC$ on K_n. Hence $F \in ACG$ on K. Likewise $F \in AC$ on K.

<u>**Theorem 5**</u>. <u>If</u> F <u>and</u> g <u>are</u> $DB_1 \cap T_2 \cap [M_*]$, g <u>is defined on</u> $[0,1]$, F <u>is defined on the range of</u> g, <u>and if both</u> F^* <u>and</u> $(F \circ g)^*$ <u>have</u> $\mathcal{C}_*PG$ - <u>major functions, then</u>

$$\mathcal{C}_*DG \int_0^1 (F^* \circ g)(x) \cdot g^*(x)dx = \mathcal{C}_*DG \int_{g(0)}^{g(1)} F^*(x)dx .$$

<u>**Proof**</u>. By Theorem 4, F is differentiable a.e. on $g([0,1])$ and $F(g(1)) - F(g(0)) = \mathcal{C}_*DG \int_{g(0)}^{g(1)} F^*(x)dx$. By Goodman's theorem of [6] or [5]. (If g is continuous a.e. on $[a,b]$, $F \in (N)$ and is defined and differentiable a.e. on the range of g, then $(F \circ g)^* = (F^* \circ g) \cdot g^*$ a.e. on $[a,b]$.), $(F \circ g)^* = (F^* \circ g) \cdot g^*$ a.e. on $[0,1]$. Since $F \in \mathcal{C}$, $F \circ g \in DB_1 \cap [M_*]$ on $[0,1]$. (See [1], page 16, Theorem 3.5 and Proposition 4.a).) $F \in \mathcal{C} \cap (N)$ implies $F \in T_2$ by [11], Theorem 7.3, page 284. Hence $F \circ g \in T_2$. (Indeed, let $A = \{z :$

$\{y : F(y) = z\}$ is nondenumerable$\}$. Then $|A| = 0$. Let $B = \{y : \{x : g(x) = y\}$ is nondenumerable$\}$. Then $|B| = 0$. Since $F \in (N)$, $|F(B)| = 0$. Let $C = \{z : \{x : F(g(x)) = z\}$ is nondenumerable$\}$. Then $C \subset A \cup F(B)$. Hence $|C| = 0$.) By Theorem 4

$$\mathcal{C}_*\mathrm{DG}\int_0^1 (F^* \cdot g)(x) \cdot g^*(x)dx = \mathcal{C}_*\mathrm{DG}\int_0^1 (F \cdot g)^*(x)dx = F(g(1)) - F(g(0)) .$$

Remark 9. If in Theorem 5, $DB_1 \cap T_2 \cap [M_*]$ is replaced by $\mathcal{C} \cap [N]$, we have Goodman's change of variables formula. (See [6] or [5].)

Theorem 6. a) <u>Let</u> $g : [0,1] \to \mathbb{R}$, $g([0,1]) \to I$, $F : I \to \mathbb{R}$, <u>where</u> I <u>is an interval</u>. <u>Let</u> $F, g \in [M] \cap B_1^*$ <u>and let</u> F, $F \cdot g \in L$. <u>If</u> F_{ap}^* (<u>resp</u>. F^*; F^*) <u>has a</u> LPG (<u>resp</u>. $L_0PG \cap \Delta_{a.e.}$; L_*PG) – <u>major function on</u> I <u>and</u> $(F \cdot g)_{ap}^*$ (<u>resp</u>. $(F \cdot g)^*$; $(F \cdot g)^*$) <u>has a</u> LPG (<u>resp</u>. $L_0PG \cap \Delta_{a.e.}$; L_*PG) – <u>major function on</u> $[0,1]$, <u>then</u>

$$\mathrm{LDG}\int_0^1 (F_{ap}^* \cdot g)(x) \cdot g_{ap}^*(x)dx = \mathrm{LDG}\int_{g(0)}^{g(1)} F_{ap}^*(x)dx \quad (\underline{\text{resp}}.$$

$$L_0\mathrm{DG}\int_0^1 (F^* \cdot g)(x) \cdot g^*(x)dx = L_C\mathrm{DG}\int_{g(0)}^{g(1)} F^*(x)dx ;$$

$$L_*\mathrm{DG}\int_0^1 (F^* \cdot g)(x) \cdot g^*(x)dx = L_*\mathrm{DG}\int_{g(0)}^{g(1)} F^*(x)dx) .$$

b) a) <u>remains true if condition</u> "F, $F \cdot g \in L$" <u>is replaced by</u> "$F,g \in D$" <u>and</u> L <u>is considered to be</u> $\mathcal{C}$.

Proof. a) By Proposition 4, $F \cdot g \in [M] \cap B_1^*$. By the first part of Theorem 3, b) and Remark 6, a), $F \in ACG$, F is approximately differentiable a.e. on I and

$$F(g(1)) - F(g(0)) = \mathrm{LDG}\int_{g(0)}^{g(1)} F_{ap}^*(x)dx .$$

By Foran's Theorem 0 of [5] (If $g : [a,b] \to \mathbb{R}$, $F \in (N)$ and F is defined on an interval containing the range of g and is apprximately differentiable a.e. on the range of g, then $(F \cdot g)_{ap}^*(x) = (F_{ap}^* \cdot g)(x) \cdot g_{ap}^*(x)$ a.e. on [a,b]), we have $(F \circ g)^*_{ap}(x) = (F^*_{ap} \circ g)(x) \cdot g^*_{ap}(x)$ a.e. on $[0,1]$. By the first part of Theorem 3, b) and Remark 6, a),

$$\mathrm{LDG}\int_0^1 (F^*_{ap} \circ g)(x) \cdot g^*_{ap}(x)dx = \mathrm{LDG}\int_0^1 (F \circ g)^*_{ap}(x)dx = F(g(1) - F(g(0)) .$$

The proof of the second and the third part is analogous to the proof of Theorem 5.

b) If $F \in C$ and $g \in DB^*_1$, then $F \circ g \in DB^*_1$. Now the proof follows using Theorem 3, b) and Remark 6, b) as in a).

<u>Remark 10</u>. In the first part of Theorem 6, a), b), "F^+_{ap} has a LPG major function" can be replaced by "F^* has a L_0PG major function" and "$(F \circ g)^*_{ap}$ has a LPG major function" can be replaced by "$(F \circ g)^*$ has a L_0PG major function".

<u>Remark 11</u>. The first part of Theorem 6, b) and Remark 10 extend Foran's change of variables formula for the Denjoy integral.

<u>An integral of Perron type for the Foran integral</u>

<u>Definition 5</u>. Let $0_+(f;E) = \sup\{f(y) - f(x) : x,y \in E, x \leq y\}$; $0_-(f;E) = \inf\{f(y) - f(x) : x,y \in E, x \leq y\}$; $0(f;E) = \max\{0_+(f;E), |0_-(f;E)|\}$. Clearly $0_-(f;E) \leq 0 \leq 0_+(f;E)$.

<u>Definition 6</u>. Given a natural number N and a set E, a function f will be said to be $\underline{A}(N)$ on E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\{I_k\}$ is a sequence of nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum |I_k| < \delta$, then there exist sets E_{kn}, $n = 1,2,\dots,N$, such that

$$\bigcup_{n=1}^{N} E_{kn} = E \cap I_k \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |0_-(f;E_{kn})| < \varepsilon .$$

Let $\bar{A}(N) = \{f : -f \in \underline{A}(N)\}$. If $0_-(f;E_{kn})$ is replaced by $0(f;E_{kn})$ we obtain a condition which can be seen to be equivalent to Foran's condition $A(N)$ on E. Clearly $\underline{A}(1) = \underline{AC}$.

<u>Definition 7</u>. A function F is said to be $\underline{A}'(N)$ on a set E, if $F = F_1 + F_2$, $F_1 \in A(N)$ and $F_2 \in \underline{AC}$ on E.

Definition 8. Given a natural number N and a set Q, a function f will be said to be $\underline{E}(N)$ on Q if for every $S \subset Q$, $|S| = 0$, and for every $\varepsilon > 0$ there exist a sequence of nonoverlapping intervals $\{I_k\}$ and a sequence of sets $\{S_{kn}\}$, $n = 1,2,\dots,N$ such that

$$S \subset \bigcup_k I_k\,, \quad S \cap I_k = \bigcup_{n=1}^{N} S_{kn} \quad \text{and}$$

$$N \cdot \sum_{k=1}^{\infty} |I_k| + \sum_{k=1}^{\infty} \sum_{n=1}^{N} |O_{-}(f;S_{kn})| < \varepsilon .$$

Let $\overline{E}(N) = \{f : -f \in \underline{E}(N)\}$. If $O_{-}(f;S_{kn})$ is replaced by $O(f;S_{kn})$, we obtain a condition which can be seen to be equivalent to condition $E(N)$ on Q.

Definition 9. Let $\overline{\mathcal{F}}$ (resp. $\overline{\mathcal{F}}'$; $\overline{\mathcal{E}}$) be the class of all functions f defined on a closed interval I for which there exist a sequence of sets $\{Q_n\}$ and natural numbers $\{N_n\}$ such that $I = \cup Q_n$ and f is $\overline{A}(N_n)$ (resp. $\overline{A}'(N_n)$; $\overline{E}(N_n)$) on Q_n. Let $\underline{\mathcal{F}} = \{f : -f \in \overline{\mathcal{F}}\}$; $\underline{\mathcal{F}}' = \{f : -f \in \overline{\mathcal{F}}'\}$; $\underline{\mathcal{E}} = \{f : -f \in \overline{\mathcal{E}}\}$.

Definition 10. A function M is said to be a LFP (resp. LF'P) - major function for a function f on $[0,1]$ if: 1° $M(0) = 0$; 2° $M \in uL$ on $[0,1]$; 3° $\ell M'_{ap}(x) \geq f(x)$ a.e. on $[0,1]$; 4° $M \in \underline{\mathcal{F}} \cap B_1^*$ (resp. $M \in \underline{\mathcal{F}}' \cap B_1^*$). m is a LFP (resp. LF'P) - minor function for f if $-m$ is a LFP (resp. LF'P) - major function for $-f$ on $[0,1]$.

A function f is LFP (resp. LF'P) integrable on $[0,1]$ if: (i) f has LFP (resp. LF'P) major and minor functions on $[0,1]$; (ii) for each $\varepsilon > 0$ there exist a LFP (resp. LF'P) major function M and a LFP (resp. LF'P) minor function m such that $M(x) - m(x) \leq \varepsilon$, $x \in [0,1]$. Then LFP (resp. LF'P) $\int_0^1 f(x)dx = \inf_M \{M(1)\} = \sup_m \{m(1)\}$.

Definition 11. A function f is said to be LF integrable on $[0,1]$ if there exists a function $F \in L \cap B_1^* \cap \mathcal{F}$ such that $F'_{ap}(x) = f(x)$ a.e. on $[0,1]$. In this case the LF integral of f over $[0,1]$ is defined to be $F(1) - F(0)$.

Let C be the Cantor ternary set. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$. Let $\varphi(x)$ be the Cantor ternary function.

Example 1. Let $F : [0,1] \to \mathbb{R}$ be a continuous function such that $F(x) = (1/2) \cdot \sum_{k=0}^{\infty} \sum_{i=j_k+1}^{j_{k+1}-1} c_i(x)/2^{i-k}$, $x \in C$ and $F(x)$ is linear on each interval contiguous to C, where $\{j_k\}$ is an increasing sequence of natural numbers, $j_0 = 0$, $(1/2) \cdot (1/3^{j_k}) \geq 1/2^{j_{k+1}-k}$, for each k. Then: a) F is $\underline{A}(2)$ on C; b) $F \notin T_2$ on C; c) $F \notin \mathcal{B}$ on C.

Proof. a) Let $I \subset [0,1]$ be a closed interval with endpoints in C, and let n be the natural number such that $1/3^{n+1} \leq |I| < 1/3^n$. Let k be the natural number such that $j_k \leq n < j_{k+1}$. Since $|I| < 1/3^n$, there exist $c_1, c_2, \ldots, c_n \in \{0,2\}$ such that for each $x \in I \cap C$, $c_i(x) = c_i$, $i = 1,2,\ldots,n$. Let $a = \sum_{i=1}^{j_k} c_i/3^i$ and $b = a + 1/3^{j_k}$. Then $I \subset [a,b]$. Let $E_1 = \{x \in [a,b] \cap C : c_{j_{k+1}}(x) = 0\}$; $E_2 = \{x \in [a,b] \cap C : c_{j_{k+1}}(x) = 2\}$. Let $x,y \in E_1$, $x < y$. Then we have three situations: 1° $y - x > 1/3^{j_{k+2}-1}$; 2° $y - x = 1/3^{j_{k+2}-1}$; 3° $y - x < 1/3^{j_{k+2}-1}$.

1° Let $j_k + 1 \leq i_0 \leq j_{k+2} - 1$ such that $c_i(y) = c_i(x) = c_i$, $i \leq i_0 - 1$; $c_{i_0}(x) = 0$; $c_{i_0}(y) = 2$. Clearly $i_0 \neq j_{k+1}$. Let $a_1 = a + \sum_{i=j_k+1}^{i_0-1} c_i/3^i$. Then $x = a_1 + \sum_{i=i_0+1}^{\infty} c_i(x)$ and $y = a_1 + 2/3^{i_0} + \sum_{i=i_0+1}^{\infty} c_i(y)$. We have two cases.

(i) $j_k + 1 \leq i_0 \leq j_{k+1} - 1$. Then $F(y) - F(x) \geq F(a_1 + 2/3^{i_0}) - F(a_1 + \sum_{i=i_0+1}^{\infty} 2/3^i) = F(a_1) + 1/2^{i_0-k} - F(a_1) - 1/2^{i_0-k} = 0$.

(ii) $j_{k+1} + 1 \leq i_0 \leq j_{k+2} - 1$. Then $F(y) - F(x) \geq F(a_1 + 2/3^{i_0}) - F(a_1 + \sum_{i=i_0+1}^{\infty} 2/3^i) = F(a_1) + 1/2^{i_0-k-1} - F(a_1) - 1/2^{i_0-k-1} = 0$.

Hence $F(y) - F(x) \geq 0$.

2° We have two possibilities.

(i) $x = \sum_{i=1}^{j_{k+2}-2} c_i/3^i + \sum_{i=j_{k+2}}^{\infty} 2/3^i$ and $y = \sum_{i=1}^{j_{k+2}-2} c_i/3^i + 2/3^{j_{k+2}-1}$.

Then $F(y) - F(x) = 0$.

(ii) $x = \sum_{i=1}^{j_{k+2}-2} c_i/3^i$ and $y = \sum_{i=1}^{j_{k+2}-2} c_i/3^i + \sum_{i=j_{k+2}}^{\infty} 2/3^i$.

Then $F(y) - F(x) \geq 1/2^{j_{k+2}-k-1}$. Hence $F(y) - F(x) \geq 0$.

3° Let $a_2 = a + \sum_{i=j_k+1}^{j_{k+2}-1} c_i/3^i$. Then $x = a_2 + \sum_{i=j_{k+2}}^{\infty} c_i(x)/3^i$, $y = a_2 + \sum_{i=j_{k+2}}^{\infty} c_i(y)/3^i$; $F(a_2) \leq F(x) \leq F(a_2 + \sum_{i=j_{k+2}}^{\infty} 2/3^i) = F(a_2) + 1/2^{j_{k+2}-k-1}$ and $F(a_2) \leq F(y) \leq F(a_2) + 1/2^{j_{k+2}-k-1}$. Hence $|F(y) - F(x)| \leq 1/2^{j_{k+2}-k-1}$ and $F(y_1) - F(x_1) = -1/2^{j_{k+2}-k-1}$, where $x_1 = a_2 + \sum_{i=j_{k+2}+1}^{\infty} 2/3^i$ and $y_1 = a_2 + 2/3^{j_{k+2}}$. By 1°, 2° and 3°, $|0_-(F;E_1)| = 1/2^{j_{k+2}-k-1} \leq (1/2)\cdot(1/3^{j_{k+1}}) \leq (1/2)\cdot(1/3^{n+1}) \leq |I|/2$. Analogously $|0_-(F;E_2)| \leq |I|/2$. Hence $F \in \underline{A}(2)$ on C.

b) Let $y \in [0,1]$. Then y is uniquely represented by $\sum_{i=1}^{\infty} y_i/2^i$ where we always take the infinite representation. Let $c_i = 2y_{i-k}$, for $j_k \leq i \leq j_{k+2} - 2$. Clearly $j_k - k \leq i - k \leq j_{k+1} - k - 2$. Let $C_y = \{x \in C : c_i(x) = C_i,\ j_k \leq i \leq j_{k+1} - 2\}$. Clearly C_y is a perfect set and $F^{-1}(y) = C_y$. Hence $F \notin T_2$ on C.

c) Since $F \notin T_2$, $F \notin \mathcal{B}$ by [4], Theorem 1,f).

<u>Theorem 7</u>. a) $\overline{A}(N) \cap \mathcal{C} \subset \overline{\overline{AC}}$ <u>on</u> $[0,1]$; b) $\overline{\mathcal{F}} \cap \underline{\mathcal{F}} = \mathcal{F}$ <u>on</u> $[0,1]$; c) $\overline{\mathcal{F}} \cap \underline{\mathcal{E}} \subset \mathcal{E}$ <u>on</u> $[0,1]$; d) $\overline{\mathcal{F}} \oplus \overline{\mathcal{F}}$ <u>on</u> $[0,1]$; e) <u>If</u> $F_1 \in \overline{\mathcal{F}}$, $F_2 \in \overline{\mathcal{F}}$ <u>and</u> $0 \leq F_i(x) \leq A$, $i = 1,2$, <u>then</u> $F_1 \cdot F_2 \in \overline{\mathcal{F}}$; f) $\overline{\mathcal{F}} \oplus \overline{\mathcal{E}} = \overline{\mathcal{E}}$ <u>on</u> $[0,1]$; g) $\overline{\overline{ACG}} \subset \overline{\mathcal{F}}' \subset \overline{\mathcal{F}} \subset \overline{\mathcal{E}} \subset [\overline{M}] \subset [\overline{M}_*]$ <u>strictly</u> <u>on</u> $[0,1]$.

<u>Proof</u>. a) Let $F \in \overline{A}(N) \cap \mathcal{C}$ on $[0,1]$ and let $I \subset [0,1]$ be a closed interval such that $I = \bigcup_{n=1}^{N} E_n$. It suffices to show

$$(1) \qquad 0_+(F;I) \leq \sum_{n=1}^{N} 0_+(F;E_n) .$$

Since $F \in \mathcal{C}$, $0_+(F;E_n) = 0_+(F;\overline{E}_n)$. Let $\varepsilon > 0$ and let $a,b \in I$ such that $a < b$ and $F(b) - F(a) = 0_+(F;I)$. We may suppose $F(b) > F(a)$. Let $z_0 = b \in E_{n_1}$ for some $n_1 \in \{1,2,\dots,N\}$. Let $m_1 = \inf\{F(x) : x \in [a,z_0] \cap \overline{E}_{n_1}\}$ and $x_1 = \inf\{x \in [a,b] \cap \overline{E}_{n_1} : F(x) = m_1\}$. If

$F(a) < m_1 - \varepsilon/N$, let $z_1 \in [a,x_1]$ such that $F(z_1) \in (m_1 - \varepsilon/N, m_1)$. Clearly $z_1 \in E_{n_2} - E_{n_1}$ for some $n_2 \in \{1,2,\dots,N\}\setminus\{n_1\}$. Let $m_2 = \inf\{F(x) : x \in [a,z_1] \cap E_{n_2}\}$ and $x_2 = \inf\{x \in [a,z_1] \cap \overline{E}_{n_2} : F(x) = m_2\}$. If $F(a) < m_2 - \varepsilon/N$, we continue as above. (We have at most N steps.) Hence there exists a natural number $k \le N - 1$ such that $b = z_0 \ge x_1 > z_1 > x_2 > z_2 > \cdots > x_k > z_k > x_{k+1} \ge a$; $F(z_0) \ge F(x_1) > F(z_1) \ge F(x_2) > F(z_2) \ge \cdots \ge F(x_k) > F(z_k) \ge F(x_{k+1})$ and $F(a) \ge F(x_{k+1}) - \varepsilon/N$. Then $F(b) - F(a) \le F(b) - F(x_{k+1}) + \varepsilon/N \le (k + 1)/N + \sum_{i=0}^{k} (F(z_i) - F(x_{i+1})) \le \varepsilon + \sum_{i=1}^{N} O_+(F;E_{n_i})$. Letting $\varepsilon \to 0$, we have (1).

b) It suffices to show that if F satisfies $\overline{A}(N)$ and $\underline{A}(N')$ on a set $E \subset [0,1]$, then $F \in A(N \cdot N')$ on E. Let $\varepsilon > 0$, $\varepsilon_1 = \varepsilon/2N'$, $\varepsilon_2 = \varepsilon/2N$. For ε_1 and ε_2 let δ_1 and δ_2 be the δ given by the facts that $F \in \overline{A}(N)$ and $F \in \underline{A}(N')$ on E. Let $\delta_0 = \min\{\delta_1,\delta_2\}$. If I_k, $k = 1,2,\dots$ are nonoverlapping intervals with $I_k \cap E \neq \emptyset$ and $\sum_k |I_k| < \delta_0$, then there exist sets E_{kn}, $n = 1,2,\dots,N$, $E \cap I_k = \bigcup_{n=1}^{N} E_{kn}$ and sets E'_{kn}, $n' = 1,2,\dots,N'$, $E \cap I_k = \bigcup_{n'=1}^{N'} E'_{kn'}$, such that $\sum_k \sum_{n=1}^{N} O_+(F;E_{kn}) < \varepsilon_1$ and $\sum_k \sum_{n'=1}^{N'} |O_-(F;E'_{kn})| < \varepsilon_2$. Then $\sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} O(F;E_{kn} \cap E'_{kn'}) \le \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} (O_+(F;E_{kn}) + |O_-(F;E'_{kn})|) \le \varepsilon$. Hence $F \in A(N \cdot N')$ on E.

c) It suffices to show that if $F \in \overline{A}(N) \cap \underline{E}(N')$ on a set $Q \subset [0,1]$, then $F \in E(N \cdot N')$ on Q. Let $\varepsilon > 0$, $\varepsilon_1 = \varepsilon/(N + N')$. Let $S \subset E$, $|S| = 0$ and let δ_1 be the δ determined by ε_1 and the fact that $F \in \overline{A}(N)$ on Q. Let $\varepsilon_2 = \min\{\varepsilon_1,\delta_1\}$. Then there exist a sequence of non-overlapping intervals I_k, $I_k \cap S \neq \emptyset$, $S \subset \cup I_k$, and a sequence of sets $\{S'_{kn'}\}$, $n' = 1,\dots,N'$, such that $N' \cdot \sum_k |I_k| + \sum_k \sum_{n'=1}^{N'} |O_-(F;S'_{kn'})| < \varepsilon_2$. Since $F \in \overline{A}(N)$, there exists a sequence of set $\{S_{kn}\}$, $n = 1, 2,\dots,N$, such that $\sum_k \sum_{n=1}^{N} O_+(F;S_{kn}) < \varepsilon$. Then we have $N \cdot N' \cdot \sum_k |I_k| + \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} O(F;S_{kn} \cap S'_{kn'}) \le N \cdot N' \cdot \sum_k |I_k| + \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} (O_+(F;S_{kn}) + |O_-(F;S'_{kn'})|) \le \varepsilon$. Hence $F \in E(N \cdot N')$ on Q.

d) It suffices to show that if $F_1 \in \overline{A}(N)$ and $F_2 \in \overline{A}(N')$ on a set $E \subset [0,1]$, then $F_1 + F_2 \in \overline{A}(N \cdot N')$ on E. Let $\varepsilon > 0$, $\varepsilon_1 = \varepsilon/2N'$, $\varepsilon_2 = \varepsilon/2N$. Let δ_1 and δ_2 be the δ determined by ε_1 respectively ε_2 and the facts that $F_1 \in \overline{A}(N)$ and $F_2 \in \overline{A}(N')$. Let $\delta_0 = \min(\delta_1, \delta_2)$. If I_k, $k = 1,2,..,$ are nonoverlapping intervals, $I_k \cap E \neq \emptyset$, $\sum |I_k| < \delta_0$, then there exist sets $\{E_{kn}\}$, $n = 1,\dots,N$, $\bigcup_{n=1}^{N} E_{kn} = E \cap I_k$ and sets $\{E'_{kn}\}$, $n' = 1,\dots,N'$, $\bigcup_{n'=1}^{N'} E'_{kn} = E \cap I_k$ such that $\sum_k \sum_{n=1}^{N} 0_+(F_1;E_{kn}) < \varepsilon_1$ and $\sum_k \sum_{n'=1}^{N'} 0_+(F_2;E'_{kn}) < \varepsilon_2$. Since $0_+(F_1 + F_2;X) \leq 0_+(F_1;X) + 0_+(F_2;X)$, $X \subset E$, it follows that $\sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} 0_+(F_1 + F_2;\ E_{kn} \cap E'_{kn'}) \leq \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} 0_+(F_1;\ E_{kn} \cap E'_{kn'}) + \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} 0_+(F_2;\ E_{kn} \cap E'_{kn'}) \leq \varepsilon_1 \cdot N' + \varepsilon_2 \cdot N = \varepsilon$.

e) It suffices to show that if $F_1 \in \overline{A}(N)$ and $F_2 \in \overline{A}(N')$ on $E \subset [0,1]$, then $F_1 \cdot F_2 \in \overline{A}(N \cdot N')$. Since $0_+(F_1 \cdot F_2;\ X) = \sup\{F_1(y) \cdot F_2(y) - F_1(x) \cdot F_2(x) : x,y \in X,\ x \leq y\} = \sup\{F_2(y) \cdot (F_1(y) - F_1(x)) + F_1(x) \cdot (F_2(y) - F_2(x)) : x,y \in X,\ x \leq y\} \leq A \cdot 0_+(F_1;X) + A \cdot 0_+(F_2;X)$, $X \subset E$, the proof is similar to that of d).

f) It suffices to show that if $F_1 \in \overline{A}(N)$ and $F_2 \in \overline{E}(N')$ on $Q \subset [0,1]$, then $F_1 + F_2 \in \overline{E}(N \cdot N')$ on Q. Let $S \subset Q$, $|S| = 0$. Let $\varepsilon > 0$, $\varepsilon_1 = \varepsilon/(N + N')$. Let δ_1 be the δ determined by ε_1 and the fact that $F_1 \in \overline{A}(N)$ on Q. Let $\varepsilon_2 = \min\{\varepsilon_1, \delta_1\}$. Then there exist a sequence of non-overlapping intervals $\{I_k\}$, $I_k \cap S \neq \emptyset$, $S \subset \cup I_k$, and a sequence of sets $\{S'_{kn}\}$, $n' = 1,\dots,N'$, such that $N' \cdot \sum_k |I_k| + \sum_k \sum_{n'=1}^{N'} 0_+(F_2;\ S'_{kn'}) < \varepsilon_2$. Since $F_1 \in \overline{A}(N)$, there exists a sequence of sets $\{S_{kn}\}$, $n = 1,\dots,N$, such that $\sum_k \sum_{n=1}^{N} 0_+(F_1;S_{kn}) < \varepsilon_1$. Hence $N \cdot N' \cdot \sum_k |I_k| + \sum_k \sum_{n=1}^{N} \sum_{n'=1}^{N'} 0_+(F_1 + F_2;\ S_{kn} \cap S'_{kn'}) < N \cdot \varepsilon_2 + N' \cdot \varepsilon_1 = \varepsilon$.

g) We show that $\overline{\mathcal{E}} \subset [\overline{M}]$. Let $F \in \overline{\mathcal{E}}$ on $[0,1]$. Let $P = \overline{P} \subset [0,1]$ be such that $F|_P$ is continuous and increasing. Since $F \in \overline{\mathcal{E}}$ and $0_+(F;X) = 0(F;X)$ on a set X, if F is increasing on X, $F \in \mathcal{E}$ on P.

Hence $F \in (N)$ on P. By [11] (Theorem 6.7, page 227), $F \in AC$ on P. By Theorem 3 of [4], $F \in [\bar{M}]$.

We show that $\bar{\mathcal{F}} \subset \bar{\mathcal{E}}$. It suffices to show that if $F \in \bar{A}(N)$, then $F \in \bar{E}(N)$ on a set Q. Let $\varepsilon > 0$ and let δ be determined by ε and the fact that F satisfies $A(N)$ on Q. Let $S \subset [0,1]$, $|S| = 0$. Select a sequence of nonoverlapping intervals $\{I_k\}$ such that $S \subset \cup I_k$ and $\sum |I_k| < \min\{\varepsilon,\delta\}$. Let S_{kn}, $n = 1,\dots,N$, be sets such that $S \cap I_k = \bigcup_{n=1}^{N} S_{kn}$ and $\sum_{k=1}^{\infty} \sum_{n=1}^{N} O_+(F;S_{kn}) < \varepsilon$. Then $N \cdot \sum_{k=1}^{\infty} |I_k| + \sum_{k=1}^{\infty} \sum_{n=1}^{N} O_+(F;S_{kn}) < \varepsilon \cdot (N + 1)$. Hence $F \in \bar{E}(N)$ on Q. The other inclusions are evident. It remains to show that they are also strict. We show that $\overline{ACG}$ is strictly contained in $\bar{\mathcal{F}}'$. Let $F, G : [0,1] \longrightarrow \mathbb{R}$, $F \in \mathcal{F}$, $G \in (N)$ such that $F + G = \varphi$. (See [3], the the proof of Theorem 4, page 205.) Then $-G = F - \varphi \in \bar{\mathcal{F}}'$. Suppose on the contrary that $-G \in \overline{ACG} \subset VBG$. Since $-G \in (N)$, $G \in ACG$ ([11], Theorem 8.8, page 233). Hence $\varphi = F + G \in \mathcal{F}'$, a contradiction. Thus $\bar{\mathcal{F}}' - ACG \neq \emptyset$. We show that $\bar{\mathcal{F}}'$ is strictly contained in $\bar{\mathcal{F}}$. Clearly $\bar{\mathcal{F}}' \subset \mathcal{B} \subset T_2$. (See [4], Theorem 1,c).) Let F be the function constructed in Example 1. Then $-F \in \bar{\mathcal{F}} - T_2$. Hence $-F \notin \bar{\mathcal{F}}'$.

We show that $\bar{\mathcal{F}}$ is strictly contained in $\bar{\mathcal{E}}$. Let $F_1, F_2 : [0,1] \longrightarrow \mathbb{R}$ be the functions defined in [3] (Theorem 5,a)), $F_1 + F_2 = \varphi$, $F_1, F_2 \in \mathcal{E}$. Suppose on the contrary that $F_1 \in \bar{\mathcal{F}}$. By f) $\varphi = F_1 + F_2 \in \bar{\mathcal{E}}$. But $\bar{\mathcal{E}} \subset [\bar{M}]$. Hence we have a contradiction.

We show that $\bar{\mathcal{E}}$ is strictly contained in $[\bar{M}]$. Let $F,G : [0,1] \longrightarrow \mathbb{R}$, $F \in \mathcal{F}$, $G \in (N) \subset [\bar{M}]$, $F + G = \varphi$ ([3]). Suppose on the contrary that $G \in \bar{\mathcal{E}}$. By f) $\varphi = F + G \in \bar{\mathcal{E}} \subset [\bar{M}]$, a contradiction.

We show that $[\bar{M}]$ is strictly contained in $[\bar{M}_*]$. We consider the function g constructed in the Example of [4]. Then $g \in [M_*] = [\bar{M}_*] \cap [\underline{M}_*]$. By Theorem 3 of [4], $g \notin [\bar{M}]$.

<u>**Remark 12**</u>. Let $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = 0$, $x \in C$, $f(x) = 1$, $x \notin C$. Then f is lower semicontinuous, $f \in A(2)$ and $f \notin \underline{AC}$ on $[0,1]$. Hence we cannot give up the continuity condition in Theorem 7, a).

Lemma 1. *Let* $F_1, F_2 : [0,1] \to \mathbb{R}$, *and let* P *be a closed subset of* $[0,1]$. *If* $F_1 \in \underline{A}'(N)$ *on* P, *for some natural number* N, $F_2 \in \overline{\mathcal{E}}$ (*resp.* $\mathcal{E}$) *on* P *and if* $H = F_1 - F_2 \in VB$ *on* P, *then* $F_2|_P \in \overline{A}'(N)$ (*resp.* $A(N)$).

Proof. $F_2 = F_1 - H = f_1 - h_1 - H$, where $f_1 \in A(N)$ and $h_1 \in \underline{AC}$ on P. Then $F_2 - f_1 = h_1 - H$ is $VB \cap \overline{\mathcal{E}}$ (resp. $VB \cap \mathcal{E}$) on P, by Theorem 7 (resp. [3], Theorem 5,c), page 208). Hence $F_2 - f_1$ is $\overline{AC}$ (resp. AC) on P by Theorem 7 (resp. [3], Theorem 5,c), page 208) and $F_2 \in \overline{A}'(N)$ (resp. $A(N)$) on P.

Theorem 8. *Let* $F_1, F_2 : [0,1] \to \mathbb{R}$, $F_1 \in \underline{\mathcal{F}}' \cap B_1^*$, $F_2 \in \overline{\mathcal{E}} \cap B_1^*$ (*resp.* $F_2 \in \mathcal{E} \cap B_1^*$), $F_1 - F_2 \in VBG$ *on* $[0,1]$. *Then* $F_2 \in \mathcal{F}' \cap B_1^*$ (*resp.* $F_2 \in \mathcal{F} \cap B_1^*$) *on* $[0,1]$.

Proof. Let $\{P_k\}$ be a sequence of closed subsets of $[0,1]$ such that $\cup P_k = [0,1]$, $F_1|P_k \in \mathcal{C} \cap \underline{A}'(N_k)$, $F_2|P_k \in \mathcal{C} \cap \overline{\mathcal{E}}$ and $F_1 - F_2 \in VB$. By Lemma 1, $F_2|P_k \in \overline{A}'(N_k)$ (resp. $A(N_k)$). Hence $F_2 \in \mathcal{F}' \cap B_1^*$ (resp. $F_2 \in \mathcal{F} \cap B_1^*$) on $[0,1]$.

Lemma 2. *Let* $M_n : [0,1] \to \mathbb{R}$ *and let* P *be a closed subset of* $[0,1]$. *Let* $F : [0,1] \to \mathbb{R}$ *such that* $H_n(x) = M_n(x) - F(x)$ *is increasing on* P *and* $H_n \to 0$ [unif] *on* P. *If there exists a natural number* N *such that* $M_n \in \underline{A}(N)$ *on* P, *then* $F \in \underline{A}(N)$ *on* P.

Proof. Let $\varepsilon > 0$ and let n be a natural number such that $H_n(x) \le \varepsilon/2N$, $x \in P$. Let $\delta(n,\varepsilon)$ be the δ determined by $\varepsilon/2$ and the fact that $M_n \in \underline{A}(N)$ on P. Let $\{I_k\}$ be a sequence of nonoverlapping intervals, $I_k \cap P \ne \emptyset$, $\sum |I_k| < \delta(n,\varepsilon)$. Let E_{kj}, $j = 1,\dots,N$ be sets such that

$$(2) \qquad \bigcup_{j=1}^{N} E_{kj} = I_k \cap P \quad \text{and} \quad \sum_k \sum_{j=1}^{N} |0-(M_n;E_{kj})| < \varepsilon/2 .$$

Let $a_{kj}, b_{kj} \in E_{kj}$, $a_{kj} \le b_{kj}$. Then $\sum_k (F(b_{kj}) - F(a_{kj})) = \sum_k (M_n(b_{kj}) - M_n(a_{kj})) - \sum_k (H_n(b_{kj}) - H_n(a_{kj})) \ge \sum_k (M_n(b_{kj}) - M_n(a_{kj})) - \varepsilon/2N$. By (2) we have $\sum_k \sum_{j=1}^{N} (F(b_{kj}) - F(a_{kj})) \ge -\varepsilon/2 - \varepsilon/2 = -\varepsilon$. Hence $F \in \underline{A}(N)$ on P.

Theorem 9. Let uL be an upper semilinear space contained in uCM which is closed under uniform convergence. Then the LF integral is equivalent with the LF'P integral on [0,1].

Proof. If f is LF integrable on [0,1] and F is an indefinite LF integral of f on [0,1], then $F(x) - F(0)$ serves as both, a LF'P - major and minor function for f on [0,1]. Hence f is LF'P integrable with F an indefinite LF'P integral. Conversely, let f be an LF'P integrable function with F an indefinite LF'P integral. Let M_i and m_j denote respectively LF'P - major and minor functions for f on [0,1] with the following properties:

$$(3) \qquad M_i(a) = m_i(a) = 0; \quad M_i(x) - m_i(x) \leq 1/i, \quad x \in [0,1];$$

$$(4) \qquad (M_i)'_{ap}(x) \geq f(x) \geq (m_j)'_{ap}(x) \quad \text{a.e. on } [0,1] .$$

Clearly $F(x) = \sup_j\{m_j(x)\} = \inf_i\{M_i(x)\}$. By Theorem 11 of [4], $M_i - m_j$ is increasing, and by (3), $M_i \to F$ [unif], $m_j \to F$ [unif]. Hence $M_i - F$ and $F - m_j$ are increasing on [0,1]. Let $\{P_k\}$ be a sequence of closed subsets of [0,1] such that $M_i|P_k \in \mathcal{C} \cap \underline{A}'(N_{ik})$ and $m_j|P_k \in \mathcal{C} \cap \overline{A}'(N'_{jk})$. Let $N_k = N_{1k}$. Then $M_1|P_k \in \underline{A}'(N_k)$. Since $m_j \in \overline{\mathcal{F}}' \subset \overline{\mathcal{E}}$ on P_j, for each j, by Lemma 1, $m_j \in \overline{A}'(N_k)$ and $M_i \in \underline{A}(N_k)$ on P_k, for each i and j. By Lemma 2, $F \in \underline{A}(N_k) \cap \overline{A}(N_k) = A(N_k \cdot N_k)$ on P. Hence $F \in \mathcal{F}' \cap B_1^*$ on [0,1]. Since $M_i \to F$ [unif], $m_j \to F$ [unif] and uL is closed under uniform convergence, it follows that $F \in L$. Since $M_i - F$, $F - m_i$ and $M_i - m_i$ are increasing on [0,1], $(M_i)'_{ap}(x) \geq F'_{ap}(x) \geq (m_i)'_{ap}(x)$. By (3) and (4) $\int_0^1 |F'_{ap}(x) - f(x)|dx \leq \int_0^1 |(M_i)'_{ap}(x) - (m_i)'_{ap}(x)|dx \leq M_i(1) - m_i(1) \leq 1/i$. Hence $\int_0^1 |F'_{ap}(x) - f(x)|dx = 0$. Therefore $F'_{ap}(x) = f(x)$ a.e. on [0,1].

Example 2. Let $\{j_k\}$ be an increasing sequence of natural numbers and let p be a natural number such that: $j_0 = 0$ and $2^{j_{2k+4}-p} \geq 2^{j_{2k+4}-k} \geq 3^{j_{2k+2}-1}$, $k \geq p$. Let $F, G_p : [0,1] \to \mathbb{R}$ be two continuous functions defined as follows: $F(x) = \sum_{i=0}^{\infty} \sum_{k=j_{2i}+1}^{j_{2i+1}} c_k(x)/2^{k+1}$, $x \in C$ and F is linear on each interval contiguous to C; $G_p(x) = F(x) + (1/(2P - 1)) \cdot \varphi(x)$. Then we have

a) $G_p' = F'$ a.e. <u>on</u> $[0,1]$ <u>and</u> $G_p \to F$ [unif], $p \to \infty$;

b) $G_p \in \underline{A}(2P)$ <u>on</u> C, <u>hence</u> $G_p \in \underline{\mathcal{F}}$ <u>on</u> $[0,1]$;

c) F <u>and</u> G_p <u>belong to the class</u> $\mathcal{B}$;

d) $F \in \mathcal{E} - \mathcal{F}$ <u>on</u> $[0,1]$. <u>Moreover</u>, $F \notin \underline{\mathcal{F}}$ <u>and</u> $F \notin \overline{\mathcal{F}}$.

<u>Proof</u>. a) is evident.

b) Let $\varepsilon > 0$, $\delta_p = \min\{\varepsilon, 1/3^{j_{2p}+1}\}$. Let $I \subset [0,1]$ be an interval with endpoints in C such that $|I| < 1/3^{j_{2p}}$. Let n be a natural number such that $1/3^{n+1} \leq |I| < 1/3^n$, and let k be a natural number such that $j_{2k} \leq n < j_{2k+2}$. Clearly $k \geq p$. Since $|I| < 1/3^n$, there exist $c_1, c_2, \ldots, c_n \in \{0,2\}$ such that $c_i(x) = c_i$, $i = 1,2,\ldots,n$, $x \in I \cap C$. Let $a = \sum_{i=1}^{j_{2k}} c_i/3^i$ and $b = a + 1/3^{j_{2k}}$. Let $d_i \in \{0,2\}$, $i = 1,2,\ldots,p$, and let $E_{d_1\ldots d_p} = \{x \in [a,b] \cap C : c_{j_{2k+2}}(x) = d_1;\ c_{j_{2k+2}-1}(x) = d_2;\ \ldots;\ c_{j_{2k+2}-p+1} = d_p\}$. Let $x,y \in E_{d_1\ldots d_p}$, $x < y$. Then we have three situations: 1) $y - x > 1/3^{j_{2k+3}}$; 2) $y - x = 1/3^{j_{2k+3}}$; 3) $y - x < 1/3^{j_{2k+3}}$.

1) Let $j_{2k} + 1 \leq i_o \leq j_{2k+3}$ such that $c_i(x) = c_i(y) = c_i$, $i \leq i_o - 1$; $c_{i_o}(x) = 0$; $c_{i_o}(y) = 2$. Clearly $i_o \in \{j_{2k+2}, j_{2k+2}-1, \ldots, j_{2k+2}-p+1\}$. Then $x = a_1 + \sum_{i=i_o+1}^{\infty} c_i(x)/3^i$, $y = a_1 + 2/3^{i_o} + \sum_{i=i_o+1}^{\infty} c_i(y)/3^i$, $a_1 = \sum_{i=1}^{i_o-1} c_i/3^i$. We have three possibilities: (i) $j_{2k} + 1 \leq i_o \leq j_{2k+1}$; (ii) $j_{2k+1} + 1 \leq i_o \leq j_{2k+2} - p$; (iii) $j_{2k+2} + 1 \leq i_o \leq j_{2k+3}$.

(i) $G_p(y) - G_p(x) = F(y) - F(x) + (\varphi(y) - \varphi(x))/(2P - 1) \geq F(y) - F(x) \geq F(a_1 + 2/3^{i_o}) - F(a_1 + \sum_{i=l_o+1}^{\infty} 2/3^i) > 0.$

(ii) $G_p(y) - G_p(x) \geq G_p(a_1 + 2/3^{i_o} + \sum_{i=j_{2k+2}-p+1}^{j_{2k+2}} (d_i - j_{2k+2} + p)/3^i) - G_p(a_1 + \sum_{i=i_o+1}^{j_{2k+2}-p} 2/3^i + \sum_{i=j_{2k+2}-p+1}^{j_{2k+2}} (d_i - j_{2k+2}-p)/3^i + \sum_{i=j_{2k+2}}^{\infty} 2/3^i) = G_p(2/3^{i_o}) - G_p(\sum_{i=i_o+1}^{j_{2k+2}-p} 2/3^i + \sum_{i=j_{2k+2}+1}^{\infty} 2/3^i) = F(2/3^{i_o}) - F(\sum_{i=i_o+1}^{j_{2k+2}-p} 2/3^i) -$

$F(\sum_{i=j_{2k+2}+1}^{\infty} 2/3^i) + (1/(2P-1)) \cdot [(\varphi(2/3^{i_0}) - \varphi(\sum_{i=i_0+1}^{j_{2k+2}-P} 2/3^i) - \varphi(\sum_{i=j_{2k+2}+1}^{\infty} 2/3^i)) = -F(\sum_{i=j_{2k+2}+1}^{\infty} 2/3^i) + (1/(2P-1)) \cdot [(1/2^{j_{2k+2}-P} - 1/2^{j_{2k+2}})] > -1/2^{j_{2k+2}} + (1/(2^P-1)).\ (2^P-1)/2^{j_{2k+2}} = 0.$

(iii) $G_p(y) - G_p(x) > 0$ (the proof is analogous to that of (i)).

2) Let $a_2 = \sum_{i=1}^{j_{2k+3}-1} c_i/3^i$. We have two possibilities:

(i) $x = a_2 + 1/3^{j_{2k+3}}$, $y = a_2 + 2/3^{j_{2k+3}}$. Then $G_p(y) - G_p(x) = F(y) - F(x) > 0$.

(ii) $x = a_2$, $y = a_2 + 1/3^{j_{2k+3}}$ and $G_p(y) - G_p(x) > 0$.

3) Let $i_0 \geq j_{2k+3}+1$ such that $c_{i_0}(x) = 0$, $c_{i_0}(y) = 2$, $c_i(x) = c_i(y) = c_i$, $i = 1,2,\ldots,i_0 - 1$. Let $a_3 = \sum_{i=1}^{i_0-1} c_i/3^i$. Then $G_p(y) - G_p(x) \geq G_p(a_3 + 2/3^{i_0}) - G_p(a_3 + 1/3^{i_0}) = F(2/3^{i_0}) - F(1/3^{i_0})$. Let m be a natural number such that $j_{2k+m+2}+1 \leq i_0 \leq j_{2k+m+4}$. We have two possibilities:

(i) m is even. Then $F(2/3^{i_0}) - F(1/3^{i_0}) = -F(\sum_{i=j_{2k+m+4}+1}^{\infty} 2/3^i) > -\varphi(1/3^{j_{2k+m+4}}) = -1/2^{j_{2k+m+4}}$.

(ii) m is odd. Then $F(2/3^{i_0}) - F(1/3^{i_0}) > 0$. By 1), 2), 3) it follows that $|O_-(G_p;E_{d_1\ldots d_p})| \leq 1/2^{j_{2k+4}}$. Since $|I| > 1/3^{n+1} \geq 1/3^{j_{2k+2}-1} \geq 2P/2^{j_{2k+4}}$, $k \geq p$ (by hypothesis), $\sum_{d_1} \sum_{d_2} \cdots \sum_{d_p} |O_-(G_p;E_{d_1\ldots d_p})| < |I|$. Now the proof follows by defintion.

c) See the lemma on page 198 of [3].

d) If $3^{j_i} < 2^{j_{i+1}}$, then both F and $f = \varphi - F$ belong to $\mathcal{E} - \mathcal{F}$. (See [3], page 208, the functions F_1, F_2.) Suppose on the contrary that $F \in \overline{\mathcal{F}}$. Then by Theorem 7, f), g), $F + f = \varphi \in \overline{\mathcal{E}} \subset [\overline{M}]$, a contradiction (since $\varphi \notin [\overline{M}]$). Hence $F \in \overline{\mathcal{F}}$. We show that $F \notin \underline{\mathcal{F}}$. It is sufficient to prove that $F \in \underline{A}(2^N - 1)$ for some natural number N, on no portion of C.

Let P be a portion of C. Suppose on the contrary that $F \in \underline{A}(2^N - 1)$ on P. Let $[a_0,b_0]$ be a closed interval retained in the Cantor ternary process from the qth step such that $[a_0,b_0] \cap C \subset P$. (We take the first q with this property.) Then $F \in \underline{A}(2^N - 1)$ on $[a_0,b_0] \cap C$. We may suppose that $j_{2k+1} < j_{2k+2}-N$ and $j_{2k+2}-N > q$. Let $n_i = j_{i+1} - j_i$. Then $N - n_{2k+1} < 0$. Let $I = [a,b]$ be a closed interval retained in the Cantor ternary process from the step $j_{2k+2}-N$, $I \subset [a_0,b_0]$. (We have $2^{j_{2k+2}-N-q}$ such intervals.) Then $a = \sum_{i=1}^{j_{2k+2}-N} c_i/3^i$, $b = a + 1/3^{j_{2k+2}-N}$. Let $\{E_n\}$, $n = 1,2,\ldots,2^N - 1$, be sets such that $E_n = \overline{E}_n \subset I \cap C$, $\bigcup_{n=1}^{2^N-1} E_n = I \cap C$. Then

$$(5) \qquad \sum_{n=1}^{2^N-1} |O_-(F;E_n)| > 2/2^{j_{2k+2}+2}. \quad \text{Hence } \sum_I \sum_{n=1}^{2^N-1} |O_-(F;E_n)|$$

$$> 2^{j_{2k+2}-N-q} \cdot (2/2^{j_{2k+2}+2}) = 1/2^{N+q+1}.$$

Since $|I| \cdot 2^{j_{2k+2}-N-q} \longrightarrow 0$, $k \longrightarrow \infty$, it follows that $F \notin \underline{A}(2^N-1)$ on $[a_0,b_0] \cap C$, a contradiction. Hence $F \notin \underline{A}(2^N-1)$ on P. It remains to show (5). Let $I_n = [a_n,b_n]$, $n = 1,\ldots,2^N$, $a_1 = a$, be the closed intervals retained in the Cantor ternary process from the step j_{2k+2} which are contained in I (numbered from left to right). We observe that $|I_n| = 1/3^{j_{2k+2}}$ and $F(a_n + x) = F(a + x)$, for each x belonging to $[0, 1/3^{j_{2k+2}}] \cap C$, $n = 1,2,\ldots,2^N$. Let $R_i = E_i - \bigcup_{t=1}^{i} I_t$, $i = 1,2,\ldots,2^N - 1$. Clearly there exists $i \in \{1,\ldots,2^N-1\}$ such that $b_i \in E_i$ and $R_i \neq \emptyset$. Let i_1 be the first i with this property. Let $x_i = b_i$, $i = 1,2,\ldots,i_1$. Let $m_{i_1} = \inf\{F(x) : x \in R_{i_1}\}$. Then $|O_-(F;E_{i_1})| \geq M_{i_1} - m_{i_1}$, where $M_{i_1} = F(x_{i_1})$.

a₁) If $m_{i_1} = F(a)$, then $\sum_{n=1}^{2^N-1} |O_-(F;E_n)| > |O_-(F;E_1)| \geq M_{i_1} - m_{i_1} = F(b) - F(a)$;

b₁) If $m_{i_1} > F(a)$, let $p_i^{(1)} = \sup\{x \in I_i : F(x) \leq m_{i_1}\}$, $i = i_1 + 1, i_1 + 2,\ldots,2^N$. Clearly $F(p_{i_1+1}^{(1)}) = \cdots = F(p_{2^N}^{(1)})$. Let $i_2 \in \{i_1 + 1,\ldots,2^N - 1\}$ be the first index such that $R_{i_2} \neq \emptyset$

and $p^{(1)}_{i_2} \in E_{i_2}$. Let $x_i = p^{(1)}_i$, $i = i_1 + 1,\dots,i_2$. Let $m_{i_2} = \inf\{F(x) : x \in R_{i_2}\}$. Then $|O_-(F;E_{i_2})| \geq M_{i_2} - m_{i_2}$, where $M_{i_2} = F(x_{i_2})$;

a₂) If $m_{i_2} = F(a)$, then $\sum_{n=1}^{2^N-1} |O_-(F;E_n)| \geq |O_-(F;E_{i_1})| + |O_-(F;E_{i_2})| \geq F(b) - F(a) - (m_{i_1} - M_{i_2})$.

b₂) If $m_{i_2} > F(a)$, let $p^{(2)}_i = \sup\{x \in I_i : F(x) < m_{i_2}\}$, $i = i_2 + 1,\dots,2^N$. Clearly $F(p^{(2)}_{i_2+1}) = \cdots = F(p^{(2)}_{2^N})$. Let $i_3 \in \{i_2 + 1,\dots,2^N - 1\}$, be the first index such that $R_{i_3} \neq \emptyset$ and $p^{(2)}_{i_3} \in E_{i_3}$. Let $x_i = p^{(2)}_i$, $i = i_2 + 1,\dots,i_3$. Let $m_{i_3} = \inf\{F(x) : x \in R_{i_3}\}$. Then $|O_-(F;E_{i_3})| \geq M_{i_3} - m_{i_3}$, where $M_{i_3} = F(x_{i_3})$;

a₃) . . . .

b₃) . . . .

. . . .

There exists some $j_o \in \{1,\dots,2^N - 1\}$ such that $R_{i_{j_o}} \neq \emptyset$ and $R_i = \emptyset$, $i > i_{j_o}$. Then $a_{2^N} \in R_{i_{j_o}}$. (Indeed, since $a_{2^N} \notin E_i$ for $R_i = \emptyset$, it follows that $a_{2^N} \in E_{i_o}$ with $R_{i_o} \neq \emptyset$, for some i_o. Hence $i_o = i_{j_o}$.) $a_{i_{j_o}}$) It follows that $m_{i_{j_o}} = F(a_{2^N}) = F(a)$. We have

(6)
$$\sum_{n=1}^{2^N-1} |O_-(F;E_n)| \geq \sum_{t=1}^{j_o} |O_-(F;E_{i_t})| \geq F(b) - F(a) - \sum_{t=1}^{j_o-1} (m_{i_t} - M_{i_{t+1}}) > (F(b) - F(a))/2 \geq 2/2^{j_{2k+2}+2} .$$

Hence we have (5). It remains to show (6). Let $Q = F(I \cap C) = F(I_1 \cap C) = \cdots = F(I_n \cap C)$. If $m_{i_t} \neq M_{i_{t+1}}$, $t = 1,\dots,j_o - 1$, then $(M_{i_{t+1}},m_{i_t})$ are intervals contiguous to $Q \subset [F(a),F(b)]$. Let $I'_m = [a'_m, b'_m]$, $m = 1,\dots,2^{n_{2k+2}}$, be the closed intervals contained in I_n, retained from the step j_{2k+3} in the Cantor ternary process. Then $Q = \bigcup_{m=1}^{2^{n_{2k+2}}} F(I'_m \cap C)$. We have $F(b'_m) - F(a'_m) = B$, where $B = \sum_{t=k+1}^{\infty} \sum_{m=j_{2t+2}+1}^{j_{2t+3}} 2/2^{m+1}$, and $F(a'_{m+1}) - F(b'_m) = A$, where $A = 2/2^{j_{2k+3}+1} - B$. Clearly $A > B$ and $(F(b'_m), F(a'_{m+1}))$ are intervals contiguous to $Q \subset [F(a), F(b)]$ with length A, $m = 1,2,\ldots,2^{n_{2k+2}} - 1$. Since $N < n_{2k+1}$, it follows that $j_0 - 1 < 2^N - 2 < 2^{n_{2k+2}} - 2$. Hence $2 \cdot (2^N - 2) < 2^{n_{2k+1}+1} - 4 < 2^{n_{2k+2}} - 1$ (since n_i is strictly increasing). We observe that: $F(b) - F(a) = 2^{n_{2k+2}} \cdot B + (2^{n_{2k+2}} - 1) \cdot A > 2 \cdot (2^N - 2) \cdot A$. Hence $(2^N - 2) \cdot A < (F(b) - F(a))/2$. Also, $\sup \sum_{t=1}^{j_0-1} (m_{i_t} - M_{i_t}) \leq (2^N - 2) \cdot A$ and $\sum_{n=1}^{2^N-1} |\dot{O}_-(F;E_n)| \geq F(b) - F(a) - (2^N - 2) \cdot A > (F(b) - F(a))/2$ and we have (6).

Open Problem. Clearly if f is LF integrable, then f is LFP integrable. Is Theorem 9 true if the LF'P integral is replaced by the LFP integral?

Remark 13. a) Theorem 8 does not remain true if the function F_1 is supposed to be $\underline{\mathcal{F}} \cap B_1^*$. (Each function G_p constructed in Example 2 is a counterexample, since $G_p \in (\underline{\mathcal{F}} \cap \mathcal{B}) - \underline{\mathcal{F}}'$.)

b) Let G_p be the functions constructed in Example 2. Suppose that N_p is the first natural number such that $G_p \in \underline{A}(N_p)$ on C. Clearly $N_p \leq 2^p$. By Lemma 2 it follows that the sequence $\{N_p\}$ is not bounded.

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